MATHEMATICS PAPER - IB

COORDINATE GEOMETRY (2D &3D) AND CALCULUS.

TIME: 3hrs. **Max. Marks.75**

Note: This question paper consists of three sections A, B and C.

SECTION - A

Very short answer type questions. **10X2 =20**

1. Show that the straight lines (a –b)x + (b –c) y = c – a, (b-c) x+ (c-a)y= (a – b) and (c –a)x + (a – b)y b - c are concurrent.

2. Fund the value of P, if the straight lines 3x + 7y – 1 = 0 and 7x – p y + 3 = 0 are mutually perpendicular.

3. Find the coordinates of the vertex C of ΔABC if its centroid is the origin and the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.

4. Find the constant k so that the planes x – 2y + kz = 0 and 2x + 5y – z = 0 are at right angles.

5. Compute $\underset{x \to 0}{Lt} \frac{a^x - 1}{b^x - 1} (a > 0, b > 0, b \neq 1)$.

6. If f is given by $f(x) = \begin{cases} k^2 x - k & \text{if} \quad x \geq 1 \\ 2 & \text{if} \quad x < 1 \end{cases}$ is a continuous function on R, then find the values of k.

7. $y = \log(\tan 5x)$, find $\frac{dy}{dx}$.

8. If $x^4 + y^4 - a^2 xy = 0$, then find $\frac{dy}{dx}$

9. Find approximate value of $\sqrt[3]{7.8}$

10. Verify the Rolle's theorem for the function $(x^2 - 1)(x - 2)$ on $[-1, 2]$. Find the point in the interval where the derivate vanishes.

SECTION B

Short answer type questions.

Answer any five of the following. **5 X 4 = 20**

11. Find the equation of locus of a point, the difference of whose distances from (–5, 0) and (5, 0) is 8 units.

12. When the origin is shifted to the point (2,3), the transformed equation of a curve is $x^2 + 3xy - 2y^2 + 17x - 7y - 11 = 0$. Find the original equation of the curve.

13. Line L has intercepts a and b on the axes of co-ordinates. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q on the transformed axes. Prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$.

14. Evaluate $\underset{x \to 0}{Lt} \left[\frac{\cos ax - \cos bx}{x^2} \right]$

15. find the derivative of the function $f(x) = \cos^2 x$ from first principle.

16. Show that the curves $x^2 + y^2 = 2$ and $\mathbf{3x^2 + x^2 = 4x}$ have a common tangent at the point (1, 1).

17. The volume of a cube is increasing at the rate of 8 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

SECTION C

Long answer type questions.

Answer any five of the following. **5 X 7 = 35.**

18. If p and q are lengths of the perpendiculars from the origin to the straight lines $x\sec\alpha + y\operatorname{cosec}\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, prove that $4p^2 + q^2 = a^2$.

19. Show that the area of triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $\boldsymbol{lx + my + n = 0}$ is $\dfrac{n^2\sqrt{h^2 - ab}}{\left|am^2 - 2h\ell m + b\ell^2\right|}$.

20. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straight line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

21. Find the direction cosines of two lines which are connected by the relation. $l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$.

22. If $y = x^{\tan x} + (\sin x)^{\cos x}$, find $\dfrac{dy}{dx}$.

23. Find the angle between the curves $\mathbf{x + y + 2 = 0;\ x^2 + y^2 - 10y = 0}$.

24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Model Paper-1 1B

SECTION A

1. Show that the straight lines (a –b)x + (b –c) y = c – a, (b-c) x+ (c-a)y= (a – b) and (c –a)x + (a – b)y b - c are concurrent.

Sol. Equations of the given lines are

L_1=(a – b) x + (b – c) y - c + a=0 --- (1)

L_2=(b – c) x + (c – a) y - a + b=0 --- (2)

L_3=(c – a) x + (a – b) y - b + c=0 --- (3)

If three lines L_1, L_2, L_3 are concurrent , then there exists non zero real numbers $\lambda_1, \lambda_2, \lambda_3$, such that $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$.

Let $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, then $1.L_1 + 1.L_2 + 1.L_3 = 0$

Hence the given lines are concurrent.

2. Fund the value of P, if the straight lines 3x + 7y – 1 = 0 and 7x – p y + 3 = 0 are mutually perpendicular.

Sol. Given lines are 3x + 7y – 1 = 0, 7x – p y + 3 = 0

lines are perpendicular

$$\Rightarrow a_1 a_2 + b_1 b_2 = 0$$
$$\Rightarrow 3.7 + 7(-p) = 0 \Rightarrow 7p = 21 \Rightarrow p = 3$$

3. Find the coordinates of the vertex C of ΔABC if its centroid is the origin and the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.

Sol. A(1, 1, 1), B(–2, 4, 1) and (x, y, z) are the vertices of ΔABC.

G is the centroid of ΔABC

Coordinates of G are

$$\left(\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3}\right) = (0,0,0)$$

$$\frac{x-1}{3} = 0, \frac{y+5}{3} = 0, \frac{z+2}{3} = 0$$

$$x-1=0,\ y+5=0,\ z+2=0$$
$$x=1,\ y=-5,\ z=-2$$

$\therefore$ Coordinates of c are (1, –5, –2).

4. Find the constant k so that the planes x – 2y + kz = 0 and 2x + 5y – z = 0 are at right angles.

Sol. Equations of the given planes are

x – 2y + kz = 0 and 2x + 5y – z = 0

since these planes are perpendicular , therefore

$$1\cdot 2 - 2\cdot 5 + k(-1) = 0$$
$$2 - 10 = k \Rightarrow k = -8$$

5. Compute $\underset{x\to 0}{Lt} \dfrac{a^x - 1}{b^x - 1} (a > 0, b > 0, b \neq 1)$.

Sol : For $x \neq 0, \dfrac{a^x - 1}{b^x - 1} = \dfrac{\left[\dfrac{a^x - 1}{x}\right]}{\left[\dfrac{b^x - 1}{x}\right]}$

$$\underset{x\to 0}{Lt}\frac{a^x-1}{b^x-1}=\frac{\underset{x\to 0}{Lt}\frac{a^x-1}{x}}{\underset{x\to 0}{Lt}\frac{b^x-1}{x}}=\frac{\log_e^a}{\log_e^b}$$

6. If f is given by $f(x)=\begin{cases}k^2x-k & \text{if} \quad x\ge 1\\ 2 & \text{if} \quad x<1\end{cases}$ is a continuous function on R, then find the values of k.

Sol : $\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt} 2=2$

$\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt}\left(k x^2-k\right)=k^2-k$ Given $f(x)$ is continuous at $x=0$

$\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1+}{Lt} f(x)\; 2=k^2-k$

Given f is continuous on R, hence it is continuous at x=1.

Therefore L.L =R.L

⇨ $k^2-k-2=0$

$\Rightarrow (k-2)(k+1)=0 \quad \Rightarrow k=2 \text{ or } -1$

7. $y=\log(\tan 5x)$, find $\frac{dy}{dx}$.

Sol:

$$\frac{dy}{dx}=\frac{d}{dx}(\log\tan 5x)=\frac{1}{\tan 5x}\frac{d}{dx}(\tan 5x)$$

$$=\frac{5\sec^2 5x}{\tan 5x}=5.\frac{1}{\cos^2 5x.\frac{\sin 5x}{\cos 5x}}$$

$$=\frac{10}{2\sin 5x.\cos 5x}$$

$$=\frac{10}{\sin 10x}=10.\operatorname{cosec}10x$$

8. If $x^4+y^4-a^2xy=0$,then find $\frac{dy}{dx}$

Sol: Differentiate w. r. to x

$$\frac{d}{dx}\left(x^4 + y^4 - a^2\ xy\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2\left(x.\frac{dy}{dx} + y.1\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2 x\frac{dy}{dx} - a^2\ y = 0$$

$$\left(4y^3 - a^2x\right)\frac{dy}{dx} = a^2y - 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2y - 4x^3}{4y^3 - a^2x}$$

9. Find approximate value of $\sqrt[3]{7.8}$

Sol: Let x = 8, $\Delta x = -0.2, f(x) = \sqrt[3]{x}$

$$f(x + \delta x) = f(x) + f^1(x)\ \delta x$$

$$= \sqrt[3]{x} + \frac{1}{3}x^{-\frac{2}{3}}.\Delta x = \sqrt[3]{8} + \frac{1}{3.8^{\frac{2}{3}}}(-0.2)$$

$$= 2 - 0.0166 = 1.9834$$

10. Verify the Rolle's theorem for the function $(x^2 - 1)(x - 2)$ on [–1, 2]. Find the point in the interval where the derivate vanishes.

Sol. Let $f(x) = (x^2 - 1)(x - 2) = x^3 - 2x^2 - x + 2$

f is continuous on [–1, 2]

since f(–1) = f(2) = 0 and

f is differentiable on [–1, 2]

$\therefore$ By Rolle's theorem $\exists c \in (-1, 2)$

Let $f'(c) = 0$

$f'(x) = 3x^2 - 4x - 1$

$3c^2 - 4c - 1 = 0$

$$c = \frac{4 \pm \sqrt{16+12}}{6} = \frac{4 \pm \sqrt{28}}{6}$$

$$\Rightarrow c = \frac{2 \pm \sqrt{7}}{3}$$

SECTION B

11. Find the equation of locus of a point, the difference of whose distances from (–5, 0) and (5, 0) is 8 units.

Sol. Given points are A(5, 0), B(–5, 0)

Let P(x, y) be any point in the locus

Given $|PA - PB| = 8$

$\Rightarrow PA - PB = \pm 8$

$\Rightarrow PA = \pm 8 + PB$

Squaring on both sides

$PA^2 = 64 + PB^2 \pm 16PB$

$\Rightarrow (x-5)^2 + y^2 - (x+5)^2 - y^2 - 64 = \pm 16PB$

$-4 \cdot 5 \cdot x - 64 = \pm 16PB$

$-5x - 16 = \pm 4PB$

Squaring on both sides

$25x^2 + 256 + 160x = 16(PB)^2$

$= 16[(x+5)^2 + y^2]$

$= 16x^2 + 400 + 160x + 16y^2$

$9x^2 - 16y^2 = 144$

Dividing with 144, locus of P is

$$\frac{9x^2}{144}-\frac{16y^2}{144}=1\Rightarrow\frac{x^2}{16}-\frac{y^2}{9}=1$$

12. When the origin is shifted to the point (2,3), the transformed equation of a curve is $x^2+3xy-2y^2+17x-7y-11=0$. Find the original equation of the curve.

Sol. New origin =(2,3) = (h,k)

Equations of transformation are

$X=x+h,\ y=Y+k\ \rightarrow\ X=x-h=x-2, Y=y-k=y-3$

Transformed equation is

$x^2+3xy-2y^2+17x-7y-11=0$ (here x, y can be treated as upper case letters)

Original equation is

$$(x-2)^2+3(x-2)(y-3)-2(y-3)^2+17(x-2)-7(y-3)-11=0$$

$$x^2+4x+4+3xy-9x-6y+18-2y^2+12y-18+17x-34-7y+21-11=0$$

Therefore, original equation is $x^2+3xy-2y^2+4x-y-20=0$

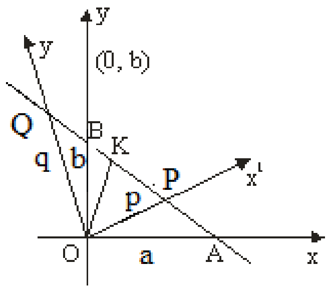
13. Line L has intercepts a and b on the axes of co-ordinates. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q on the transformed axes. Prove that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$.

Sol. Equation of the line in the old system in intercept form is $\frac{x}{a}+\frac{y}{b}=1\Rightarrow\frac{x}{a}+\frac{y}{b}-1=0$

Length of the perpendicular form origin $=\frac{|0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$ ---(1)

Equation of the line in the new system in intercept form is $\frac{x}{p}+\frac{y}{q}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}-1=0$

Length of the perpendicular $=\frac{|0+0-1|}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}$ form origin ---(2)



Since the position of origin and the given line remain unchanged ,perpendicular distances in both the systems are same.

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}} \Rightarrow \frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}=\frac{1}{\left(\frac{1}{p^2}+\frac{1}{q^2}\right)}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

14. $\underset{x\to 0}{Lt}\left[\frac{\cos ax-\cos bx}{x^2}\right]$

Sol : $\underset{x\to 0}{Lt}\frac{\cos ax-\cos bx}{x^2}=\underset{x\to 0}{Lt}\frac{2\sin\frac{(a+b)x}{2}.\sin\frac{(b-a)x}{2}}{x^2}$

$$= 2.\ \underset{x\to 0}{Lt}\ \frac{\sin(b+a)\frac{x}{2}}{x}.\ \underset{x\to 0}{Lt}\ \frac{\sin(b-a)\frac{x}{2}}{x}$$

$$= 2.\ \underset{x\to 0}{Lt}\ \frac{\sin(b+a)\frac{x}{2}}{(b+a)\frac{x}{2}} \times \frac{(b+a)}{2}$$

$$.\ \underset{x\to 0}{Lt}\ \frac{\sin(b-a)\frac{x}{2}}{(b-a)\frac{x}{2}} \times \frac{(b-a)}{2}$$

$$= 2.\left(\frac{b+a}{2}\right)\left(\frac{b-a}{2}\right) = \frac{1}{2}\left(b^2-a^2\right)$$

15. $f(x)=\cos^2 x$

Sol: $f^1(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$f^1(x) = \underset{h\to 0}{Lt}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-\left(\cos^2 x-\cos^2(x+h)\right)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-\sin(x+h+x)\sin(x+h-x)}{h}$$

$$f'(x) = \underset{h\to 0}{Lt} -\sin(2x+h)\ \underset{h\to 0}{Lt}.\frac{\sin h}{h}$$

$$= -\sin 2x.1 = -\sin 2x$$

16. Show that the curves $x^2+y^2= 2$ and $\mathbf{3x^2 + x^2 = 4x}$ have a common tangent at the point (1, 1).

Sol: Equation of the first curve is $x^2+y^2=2$

Differentiating w, r, to x

$$\Rightarrow \quad 2x+2y\frac{dy}{dx}=0 \Rightarrow 2y\frac{dy}{dx}=-2x \quad \Rightarrow \frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$$

At p (1, 1) slope of the tangent = - $\frac{-1}{1} = -1$

Equation of the second curve is $3x^2 + y^2 = 4y$.

Differentiating w. r. to x, $6x + 2y.\frac{dy}{dx} = 4 \quad \Rightarrow 2y.\frac{dy}{dx} = 4 - 6x$

$\Rightarrow \frac{dy}{dx} = \frac{4-6x}{2y} = \frac{2y-3x}{y}$

At p(1, 1) slope of the tangent = $\frac{2-3}{1} \quad = -\frac{1}{1} = -1$

The slope of the tangents to both the curves at (1, 1) are same and pass through the same point (1, 1)

$\therefore$ The given curves have a common tangent p (1, 1)

17. The volume of a cube is increasing at the rate of 8 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

Sol. Suppose 'a' is the edge of the cube and v be the volume of the cube.

$v = a^3 \qquad \ldots(1)$

given $\frac{dv}{dt} = 8cm^3 / sec$

a = 12 cm

Surface area of cube S = $6a^2$

$\frac{ds}{dt} = 12a\frac{da}{dt} \qquad \ldots(2)$

From (1), $\frac{dv}{dt} = 3a^2\frac{da}{dt}$

$8 = 3(144)\frac{da}{dt}$

$$\frac{da}{dt} = \frac{8}{3(144)} cm/s$$

$$\frac{ds}{dt} = 12a\frac{da}{dt}$$

$$= 12(12)\frac{8}{3(144)} = 144 \times \frac{8}{3(144)} = \frac{8}{3} cm^2/s$$

SECTION-C

18. If p and q are lengths of the perpendiculars from the origin to the straight lines

$x\sec\alpha + y\operatorname{cosec}\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, prove that $4p^2 + q^2 = a^2$.

Sol: Equation of AB is $x\sec\alpha + y\operatorname{cosec}\alpha = a$

$$\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} = a$$

$$x\sin\alpha + y\cos\alpha = a\sin\alpha\cos\alpha$$

$$x\sin\alpha + y\cos\alpha - a\sin\alpha\cos\alpha = 0$$

p = length of the perpendicular from O on AB = $\frac{|0+0-a\sin\alpha\cos\alpha|}{\sqrt{\sin^2\alpha + \cos^2\alpha}}$

$$= a\sin\alpha.\cos\alpha = a.\frac{\sin 2\alpha}{2} \Rightarrow$$

$2p = a\sin 2\alpha$ ---(1)

Equation of CD is $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$

$$x\cos\alpha - y\sin\alpha - a\cos 2\alpha = 0$$

q = Length of the perpendicular from O on CD $\frac{|0+0-a\cos 2\alpha|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = a\cos 2\alpha$ ---(2)

Squaring and adding (1) and (2)

$$4p^2 + q^2 = a^2\sin^2 2\alpha + a^2\cos^2 2\alpha$$

$$= a^2\left(\sin^2 2\alpha + \cos^2 2\alpha\right) = a^2.1 = a^2$$

19. The area of triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $\mathbf{lx + my + n = 0}$ is

$$\frac{n^2\sqrt{h^2-ab}}{\left|am^2-2h\ell m+b\ell^2\right|}$$

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1x+m_1y=0$ -- (1) and $l_2x+m_2y=0$ -- (2).

Then $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2 = b$.

The given straight line is $\mathbf{lx+my+n=0}$ -- (3) Clearly (1) and (2) intersect at the origin.

Let A be the point of intersection of (1) and (3). Then

x		y	1
m_1	0	l_1	m_1
m	n	l	m

$$\Rightarrow \frac{x}{m_1n-0}=\frac{y}{0-nl_1}=\frac{1}{l_1m-lm_1}$$

$$\Rightarrow x=\frac{m_1n}{l_1m-lm_1} \text{ and } y=\frac{-nl_1}{l_1m-lm_1}$$

$$\therefore A=\left(\frac{m_1n}{l_1m-lm_1},\frac{-l_1n}{l_1m-lm_1}\right)=(x_1,y_1)$$

$$B=\left(\frac{m_2n}{l_2m-lm_2},\frac{-l_2n}{l_2m-lm_2}\right)=(x_2,y_2)$$

$\therefore$The area of $\Delta OAB=\frac{1}{2}|x_1y_2-x_2y_1|$

$$=\frac{1}{2}\left|\left(\frac{m_1n}{l_1m-lm_1}\right)\left(\frac{-l_2n}{l_2m-lm_2}\right)-\left(\frac{m_2n}{l_2m-lm_2}\right)\left(\frac{-nl_1}{l_1m-lm_1}\right)\right|$$

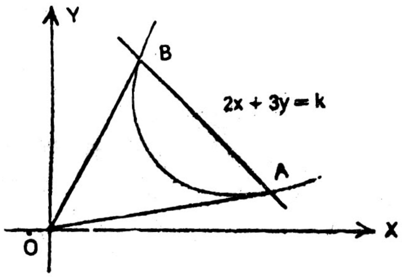
$$\frac{1}{2}\left|\frac{l_1m_2n^2-l_2m_1n^2}{(l_1m-lm_1)(l_2m-lm_2)}\right|$$

$$=\frac{n^2}{2}\left|\frac{(l_1m_2-l_2m_1)}{l_1l_2m^2-(l_1m_2+l_2m_1)lm+m_1m_2l^2}\right|=$$

$$= \frac{n^2}{2}\left|\frac{\sqrt{(l_1m_2+l_2m_1)^2-4l_1m_2l_2m_1}}{am^2-2hlm+bl^2}\right|$$

$$= \frac{n^2}{2}\frac{\sqrt{4h^2-4ab}}{|am^2-2hlm+bl^2|} = \frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$$

20. Show that the lines joining the origin to the points of intersection of the curve $x^2-xy+y^2+3x+3y-2=0$ and the straight line $x-y-\sqrt{2}=0$ are mutually perpendicular.



Sol.

Le t A,B the the points of intersection of the line and the curve.

Equation of the curve is $x^2-xy+y^2+3x+3y-2=0$ …….(1)

Equation of the line AB is $x-y-\sqrt{2}=0$

$\Rightarrow \quad x-y=\sqrt{2} \Rightarrow \frac{x-y}{\sqrt{2}}=1$ ….(2)

Homogenising, (1) with the help of (2) combined equation of OA, OB is

$x^2-xy+y^2+3x.1+3y.1-2.1^2=0$

$\Rightarrow x^2-xy+y^2+3(x+y)\frac{x-y}{\sqrt{2}}-2\frac{(x-y)^2}{2}=0$

$\Rightarrow x^2-xy+y^2+\frac{3}{\sqrt{2}}(x^2-y^2)-(x^2-2xy+y^2)=0$

$\Rightarrow x^2-xy+y^2+\frac{3}{\sqrt{2}}x^2-\frac{3}{\sqrt{2}}y^2-x^2+2xy-y^2=0$

$\Rightarrow \frac{3}{\sqrt{2}}x^2+xy-\frac{3}{\sqrt{2}}y^2=0$

$\Rightarrow$ coefficient of x^2 +coefficient of y^2 $= a + b = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}} = 0$

$\therefore$ OA, OBare perpendicular.

21. Find the direction cosines of two lines which are connected by the relation

$l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$

Sol. Given $l - 5m + 3n = 0$

$\Rightarrow l = 5m - 3n - - - - - (1)$

and $7l^2 + 5m^2 - 3n^2 = 0 - - - (2)$

Substituting the value of l in (2)

$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$

$\Rightarrow 7(25^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 175m^2 + 63n^2 - 210mn + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 - 210mn + 60n^2 = 0$

$\Rightarrow 6m^2 - 7mn + 2n^2 = 0$

$\Rightarrow (3m - 2n)(2m - n) = 0$

Case (i) : $3m_1 = 2n_1 \Rightarrow \frac{m_1}{2} = \frac{n_1}{3}$

Then $m_1 = \frac{2}{3} n_1$

From (1) $l_1 = 5m_1 - 3n_1 = \frac{10}{3} n_1 - 3n_1$

$= \frac{10n_1 - 9n_1}{3} = \frac{n_1}{3}$

$\therefore \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$

d.rs of the first line are (1, 2, 3)

Dividing with $\sqrt{1 + 4 + 9} = \sqrt{14}$

d.cs of the first line are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Case (ii) $2m_2 = n_2$

From (1) $l_2 - 5m_2 + 3n_2 = 0$

$\Rightarrow l_2 - 5m_2 + 6m_2 = 0$

$\Rightarrow -l_2 = m_2$

$\therefore \frac{l_2}{-1} = \frac{m_2}{1} = \frac{n_2}{2}$

d.rs of the second line are -1, 1, 2

Dividing with $\sqrt{1+1+4} = \sqrt{6}$

d.cs of the second line are $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

Find the direction cosines of two lines which are connected by the relation

$l - 5m + 3n = 0 \text{ and } 7l^2 + 5m^2 - 3n^2 = 0$

Sol. Given $l - 5m + 3n = 0$

$\Rightarrow l = 5m - 3n - - - - - (1)$

and $7l^2 + 5m^2 - 3n^2 = 0 - - - (2)$

Substituting the value of l in (2)

$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$

$\Rightarrow 7(25^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 175m^2 + 63n^2 - 210mn + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 - 210mn + 60n^2 = 0$

$\Rightarrow 6m^2 - 7mn + 2n^2 = 0$

$\Rightarrow (3m - 2n)(2m - n) = 0$

Case (i) : $3m_1 = 2n_1 \Rightarrow \frac{m_1}{2} = \frac{n_1}{3}$

Then $m_1 = \frac{2}{3} n_1$

From (1) $l_1 = 5m_1 - 3n_1 = \frac{10}{3} n_1 - 3n_1$

$= \frac{10n_1 - 9n_1}{3} = \frac{n_1}{3}$

$\therefore \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$

d.rs of the first line are (1, 2, 3)

Dividing with $\sqrt{1+4+9} = \sqrt{14}$

d.cs of the first line are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Case (ii) $2m_2 = n_2$

From (1) $l_2 - 5m_2 + 3n_2 = 0$

$\Rightarrow l_2 - 5m_2 + 6m_2 = 0$

$\Rightarrow -l_2 = m_2$

$\therefore \frac{l_2}{-1} = \frac{m_2}{1} = \frac{n_2}{2}$

d.rs of the second line are -1, 1, 2

Dividing with $\sqrt{1+1+4} = \sqrt{6}$

d.cs of the second line are $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

22. If $y = x^{\tan x} + (\sin x)^{\cos x}$, find $\frac{dy}{dx}$

Sol : Let $u = x^{\tan x}$ and $v = (\sin x)^{\cos x}$

$\log u = \log x^{\tan x} = (\tan x)\log x$

$\frac{1}{u}.\frac{dy}{dx} = \tan x \frac{1}{x} + (\log x)\sec^2 x.$

$\frac{du}{dx} = u\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$

$= x^{\tan x}\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$

$\log v = \log(\sin x)\cos x] = \cos x.\log\sin x$

$\frac{1}{v}.\frac{dv}{dx} = \cos x.\frac{1}{\sin x}\cos x + (\log\sin x)(-\sin x)$

$= \frac{\cos^2 x}{\sin x} - \sin x \log\left(\sin^x\right)^{\cos x}$

$\frac{dv}{dx} = v\left(\frac{\cos^2 x}{\sin x} - \sin x\log\sin x\right)$

$= (\sin x)^{\cos x}\left(\frac{\cos^2}{\sin x} - \sin x\log(\sin x)\right)$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\tan x}$

$\left(\frac{\tan x}{x} + (\log x)\left(\sec^2 x\right)\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right)$

23. Find the angle between the curves $\mathbf{x+y+2=0;\ x^2+y^2-10y=0}$

Sol: $x+y+2=0 \Rightarrow x=-(y+2)$ **----(1)**

Equation of the curve $x^2+y^2-10y=0$ **--(2)**

Solving 1 and 2, $(y+2)^2+y^2-10y=0 \quad \Rightarrow y^2+4y+4+y^2-10y=0$

$\Rightarrow 2y^2-6y+4=0 \quad \Rightarrow y^2-3y+2=0 \Rightarrow (y+1)(y-2)=0$

$\Rightarrow y=1$ or $y=2$

$x=-(y+2)$

$y=1 \Rightarrow x=-(1+2)=-3$

$y=2 \Rightarrow x=-(2+2)=-4$

The points of intersection are $P(-3,1)$ and $Q(-4,2)$,

equation of the curve is $x^2+y^2-10y=0$

Differente $x^2+y^2-10y=0$ w.r.to x.

$$\Rightarrow 2x+2y\frac{dy}{dx}-10\frac{dy}{dx}=0 \Rightarrow 2\frac{dy}{dx}(y-5)=-2x \Rightarrow \frac{dy}{dx}=-\frac{x}{y-5}$$

Equation of the line is $x+y+2=0$

Slope is $m_2 = -1$.

Case (i):

$\Rightarrow slope\ m_1=\frac{dy}{dx} at P=-\frac{-3}{1-5}=-\frac{3}{4}$ and Slope is $m_2 = -1$.

Let θ be the angle between the curves, then $\mathbf{\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|}$

$$=\left|\frac{-\frac{3}{4}+1}{1+\frac{3}{4}}\right|=\frac{1}{7} \Rightarrow \theta=\tan^{-1}\left(\frac{1}{7}\right)$$

Case (ii):

$\Rightarrow slope\ m_1=\frac{dy}{dx} at Q=-\frac{4}{2-5}=-\frac{4}{3}$ and Slope is $m_2 = -1$.

$$\Rightarrow \mathbf{\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|}=\left|\frac{-\frac{4}{3}+1}{1+\frac{4}{3}}\right|=\frac{1}{7}$$

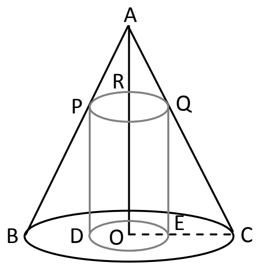
$$\Rightarrow \theta=\tan^{-1}\left(\frac{1}{7}\right)$$

24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Sol. Let O be the center of the circular base of the cone and its height be h. Let r be the radius of the circular base of the cone.

Then AO = h, OC = r

Let a cylinder with radius x(OE) be inscribed in the given cone. Let its height be u.



i.e. RO = QE = PD = u

Now the triangles AOC and QEC are similar.

Therefore, $\frac{QE}{OA} = \frac{EC}{OC}$

i.e., $\frac{u}{h} = \frac{r-x}{r}$

$\therefore u = \frac{h(r-x)}{r}$

Let S denote the curved surface area of the chosen cylinder. Then

$S = 2\pi xu$

As the cone is fixed one, the values of r and h are constants. Thus S is function of x only.

Now, $\frac{dS}{dx} = 2\pi h(r-2x)/r$ and $\frac{d^2S}{dx^2} = -\frac{4\pi h}{r}$

The stationary point of S is a root of

$\frac{dS}{dx} = 0$

i.e., $\pi(r-2x)/r = 0$

i.e., $x = \frac{r}{2}$

$\frac{d^2S}{dx^2} < 0$ for all x, therefore $\left(\frac{d^2S}{dx^2}\right)_{x=r/2} < 0$

Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is r/2.