MATHEMATICS PAPER-1A

TIME: 3hrs. **Max. Marks.75**

Note: This question paper consists of three sections A, B and C.

SECTION - A

Very short answer type questions. **10X2 =20**

1. Let ABCDEF be a regular hexagon with center O. Show that

   $\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=3\overline{AD}=6\overline{AO}$.

2. In ΔABC, if $\overline{a},\overline{b},\overline{c}$ are position vectors of the vertices A, B and C respectively, then prove that the position vector of the centroid G is $\frac{1}{3}(\overline{a}+\overline{b}+\overline{c})$.

3. For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

4. If $A-B=\frac{3\pi}{4}$, then show that (1 – tan A) (1 + tan B) = 2.

5. If $\tan\theta=\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ}$ and θ is in the third quadrant, find θ.

6. Prove that $\sin h^{-1}x=\log\left\{x+\sqrt{x^2+1}\right\}$

7. If A = $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find AA^T. Do A and A^T commute with respect to multiplication of matrices?

8. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

9. If A = {1, 2, 3, 4} and f : A → R is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range off.

10. If f = {(1, 2), (2, –3), (3, –1)} then find (i) 2 + f, (ii) $\sqrt{f}$.

SECTION B

Short answer type questions.

Answer any five of the following. **5 X 4 = 20**

11. Theorem: If A is a non-singular matrix then A is invertible and $A^{-1} = \frac{\text{AdjA}}{\det A}$.

12. In ΔABC, if O is the circumcenter and H is the orthocenter, then show that

(i) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$

(ii) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$

13. Let $\bar{a}, \bar{b}, \bar{c}$ be mutually orthogonal vectors of equal magnitudes. Prove that the vector $\bar{a} + \bar{b} + \bar{c}$ is equally inclined to each of $\bar{a}, \bar{b}, \bar{c}$, the angle of inclination being $\cos^{-1}\frac{1}{\sqrt{3}}$.

14. Prove that $\sin^2\alpha + \cos^2(\alpha + \beta) + 2\sin\alpha\sin\beta\cos(\alpha + \beta)$ is independent of α.

15. If $|\tan x| = \tan x + \frac{1}{\cos x}$ and $x \in [0, 2\pi]$ find the values of x.

16. If $Tan^{-1}x + Tan^{-1}y + Tan^{-1}z = \pi$ then prove that $x + y + z = xyz$

17. Prove that $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{bc + ca + ab - s^2}{\Delta}$

SECTION C

Long answer type questions

Answer any five of the following **5 X 7 = 35**

18. If $f : A \to B$, $g : B \to C$ are two one one onto functions then $gof : A \to C$ is also one one be onto.

19. Prove that 2.3+ 3.4 + 4.5+ …. upto n terms $\frac{n(n^2 + 6n + 11)}{3}$

20. If $|\bar{a}| = 1, |\bar{b}| = 1, |\bar{c}| = 2$ and $\bar{a} \times (\bar{a} \times \bar{c}) + \bar{b} = 0$, then find the angle between $\bar{a}$ and $\bar{c}$.

21. In a triangle ABC prove that

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$$

22. Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to

the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

23. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

24. Find the non-trivial solutions, if any, for the following equations.

$2x + 5y + 6z = 0$, $x - 3y - 8z = 0$, $3x - y - 4z = 0$

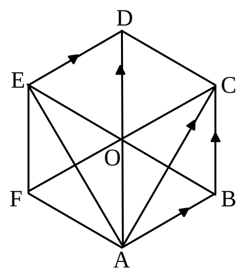
Model Paper-2 Solution

SECTION-A

1. Let ABCDEF be a regular hexagon with center O. Show that

$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=3\overline{AD}=6\overline{AO}$.

Sol.



From figure,

$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=$

$\left(\overline{AB}+\overline{AE}\right)+\overline{AD}+\left(\overline{AC}+\overline{AF}\right)$

$=\left(\overline{AE}+\overline{ED}\right)+\overline{AD}+\left(\overline{AC}+\overline{CD}\right)$

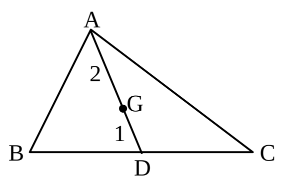
$\left(\because \overline{AB}=\overline{ED}, \overline{AF}=\overline{CD}\right)$

$=\overline{AD}+\overline{AD}+\overline{AD}=3\overline{AD}$

$=6\overline{AO}(\because$ O is the center and $\overline{OD}=\overline{AO})$

2. In ΔABC, if $\overline{a},\overline{b},\overline{c}$ are position vectors of the vertices A, B and C respectively, then prove that the positin vector of the centroid G is $\frac{1}{3}(\overline{a}+\overline{b}+\overline{c})$.

Sol.



Let G be the centroid of ΔABC and AD be the median through the vertex A. (see figure).

Then $\overline{AG}:\overline{GD}=2:1$

Since the position vector of $\bar{D}$ is $\frac{1}{2}(\bar{b}+\bar{c})$ by the Theorem 3.5.5, the position vector of G is

$$\frac{\frac{2(\bar{b}+\bar{c})}{2}+1\bar{a}}{2+1}=\frac{\bar{a}+\bar{b}+\bar{c}}{3}.$$

3. For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

Sol: $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2|\vec{b}|^2\sin^2(\vec{a},\vec{b})+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2|\vec{b}|^2\{1-\cos^2(\vec{a},\vec{b})\}+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2(\vec{a},\vec{b})+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2|\vec{b}|^2-\cancel{(\vec{a}.\vec{b})^2}+\cancel{(\vec{a}.\vec{b})^2}$

$=|\vec{a}|^2|\vec{b}|^2=R.H.S$

4. If $A-B=\frac{3\pi}{4}$, then show that (1 – tan A) (1 + tan B) = 2.

Sol. $A-B=\frac{3\pi}{4}$

$A-B=135°$

$\tan(A-B)=\tan 135°$

$=\tan(90°+45)°=-\cot 45°=-1$

$\therefore \frac{\tan A-\tan B}{1+\tan A\tan B}=-1$

$\tan A-\tan B=-(1+\tan A\tan B)$

$\tan A-\tan B=-1-\tan A\tan B$

$\tan A-\tan B+\tan A\tan B=-1$

$\tan B-\tan A-\tan A\tan B=1$ ...(1)

$\text{L.H.S.}=(1-\tan A)(1+\tan B)$

$=1+(\tan B-\tan A-\tan A\tan B)$

$=1+1$ ($\because$ from(1))

$=2=\text{R.H.S.}$

5. If $\tan\theta=\frac{\cos 11°+\sin 11°}{\cos 11°-\sin 11°}$ and θ is in the third quadrant, find θ.

Sol. $\tan\theta = \dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$

$$= \frac{\cos 11^\circ\left[1+\dfrac{\sin 11^\circ}{\cos 11^\circ}\right]}{\cos 11^\circ\left[1-\dfrac{\sin 11^\circ}{\cos 11^\circ}\right]}$$

$$= \frac{1+\tan 11^\circ}{1-\tan 11^\circ}$$

$$= \frac{\tan 45^\circ + \tan 11^\circ}{1-\tan 45^\circ \tan 11^\circ} \quad (\because \tan 45^\circ = 1)$$

$\tan\theta = \tan(45^\circ + 11^\circ)$

$= \tan 56^\circ$

$= \tan(180^\circ + 56^\circ)$

$= \tan 236^\circ$

$\theta = 236^\circ$

6. Prove that $\sin h^{-1}x = \log\left\{x+\sqrt{x^2+1}\right\}$

Let $\sin h^{-1}x = y \Rightarrow x = \sin\mathrm{h}\, y$

$$x = \frac{e^y - e^{-y}}{2} = 2x = e^4 - \frac{1}{ey} \Rightarrow 2xe^y = \left(e^y\right)^2 - 1$$

$$\left(e^y\right)^2 - 2xe^4 - 1 = 0 \Rightarrow e^4 = \frac{2x \pm \sqrt{4x^2+4}}{2} = e^y = x+\sqrt{x^2+1} \Rightarrow y = \log_e\left(x+\sqrt{x^2+1}\right)$$

$$\boxed{\therefore \sin h^{-1}x = \log_e\left\{x+\sqrt{x^2+1}\right\}}$$

7. If A = $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find AA^T. Do A and A^T commute with respect to multiplication of matrices?

Sol. $A^T = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

Since $AA^T \neq A^TA$, A and A^T do not commute with respect to multiplication of matrices.

8. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

Sol. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} [\because 1+\omega+\omega^2 = 0]$

$= 0$

9. If A = {1, 2, 3, 4} and f : A → R is a function defined by $f(x) = \frac{x^2 - x + 1}{x+1}$ then find the range of f.

Sol. Given that

$$f(x) = \frac{x^2 - x + 1}{x+1}$$

$$f(1) = \frac{1^2 - 1 + 1}{1+1} = \frac{1}{2}$$

$$f(2) = \frac{2^2 - 2 + 1}{2+1} = \frac{3}{3} = 1$$

$$f(3) = \frac{3^2 - 3 + 1}{3+1} = \frac{7}{4}$$

$$f(4) = \frac{4^2 - 4 + 1}{4+1} = \frac{13}{5}$$

$\therefore$ Range of f is $\left\{\frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5}\right\}$

10. If f = {(1, 2), (2, –3), (3, –1)} then find (i) 2 + f, (ii) $\sqrt{f}$.

Sol. Given that

$f = \{(1, 2), (2, -3), (3, -1)\}$

i) $2 + f = \{(1, 2+2), (2, -3+2), (3, -1+2)\}$
$= \{(1, 4), (2, -1), (3, 1)\}$

ii) $\sqrt{f} = \{(1, \sqrt{2})\}$

SECTION B

11. Theorem : If A is a non-singular matrix then A is invertible and $A^{-1} = \dfrac{\text{Adj}A}{\det A}$.

Sol. **Proof** : Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a non-singular matrix.

$\therefore \det A \neq 0.$

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A \cdot \text{Adj}\,A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \det A\, I$

$\therefore A \cdot \dfrac{\text{Adj}A}{\det A} = I$

Similarly we can prove that $A \cdot \dfrac{\text{Adj}A}{\det A} = I$

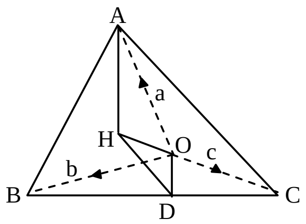
$\therefore A^{-1} = \dfrac{\text{Adj}A}{\det A}$

12. In ΔABC, if O is the circumcenter and H is the orthocenter, then show that

(i) $\overline{OA}+\overline{OB}+\overline{OC}=\overline{OH}$

(ii) $\overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$

Sol. Let D be the mid point of BC.

i) 

Take O as the origin,

Let $\overline{OA}=\bar{a}, \overline{OB}=\bar{b}$ and $\overline{OC}=\bar{c}$

(see figure)

$\overline{OD}=\dfrac{\bar{b}+\bar{c}}{2}$

$\therefore \overline{OA}+\overline{OB}+\overline{OC}=$

$\overline{OA}+2\overline{OD}=\overline{OA}+\overline{AH}=\overline{OH}$

(Observe that $\overline{AH}=2\overline{R}\cos A, \overline{OD}=\overline{R}\cos A,$

$\overline{R}$ is the circum radius of ΔABC and hence $\overline{AH}=2\overline{OD}$).

ii) $\overline{HA}+\overline{HB}+\overline{HC}=$

$\overline{HA}+2\overline{HD}=\overline{HA}+2(\overline{HO}+\overline{OD})$

$=\overline{HA}+2\overline{HO}+2\overline{OD}$

$=\overline{HA}+2\overline{HO}+\overline{AH}=2\overline{HO}$

13. Let $\bar{a},\bar{b},\bar{c}$ be mutually orthogonal vectors of equal magnitudes. Prove that the vector $\bar{a}+\bar{b}+\bar{c}$ is equally inclined to each of $\bar{a},\bar{b},\bar{c}$, the angle of inclination being $\cos^{-1}\dfrac{1}{\sqrt{3}}$.

Sol. Let $|\bar{a}|=|\bar{b}|=|\bar{c}|=\lambda$

Now, $|\bar{a}+\bar{b}+\bar{c}|^2=\bar{a}^2+\bar{b}^2+\bar{c}^2+2\Sigma\bar{a}\cdot\bar{b}$

$= 3\lambda^2 (\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 0)$

Let θ be the angle between $\bar{a}$ and $\bar{a}+\bar{b}+\bar{c}$

Then $\cos\theta = \dfrac{\bar{a}\cdot(\bar{a}+\bar{b}+\bar{c})}{|\bar{a}||\bar{a}+\bar{b}+\bar{c}|} = \dfrac{\bar{a}\cdot\bar{a}}{\lambda(\lambda\sqrt{3})} = \dfrac{1}{\sqrt{3}}$

Similarly, it can be proved that $\bar{a}+\bar{b}+\bar{c}$ inclines at an angle of $\cos^{-1}\dfrac{1}{\sqrt{3}}$ with b and c.

14. Prove that $\sin^2\alpha + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$ is independent of α.

Sol. Given expression,

$\sin^2\alpha + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= \sin^2\alpha + 1 - \sin^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + [\sin^2\alpha - \sin^2(\alpha+\beta)] + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + \sin(\alpha+\alpha+\beta)\sin(\alpha-\alpha-\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + \sin(2\alpha+\beta)\sin(-\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 - \sin(2\alpha+\beta)\sin\beta + [2\sin\alpha\cos(\alpha+\beta)]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + [\sin(\alpha+\alpha+\beta) + \sin(\alpha-\alpha-\beta)]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + [\sin(2\alpha+\beta) - \sin\beta]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + \sin(2\alpha+\beta)\sin\beta - \sin^2\beta$

$= 1 - \sin^2\beta = \cos^2\beta$

Thus the given expression is independent of α.

15. If $|\tan x| = \tan x + \dfrac{1}{\cos x}$ and $x \in [0, 2\pi]$ find the values of x

Solution:

$\cos(1)$ suppose $\tan x > 0$

$\therefore \tan x = \tan x + \dfrac{1}{\cos x} \Rightarrow \dfrac{1}{\cos x} = 0$ not possible

Case (ii) Suppose $\tan x < 0$

$\therefore -\tan x = \tan x + \dfrac{1}{\cos x} \Rightarrow -2\tan x = \dfrac{1}{\cos x}$

$$-2\sin x = 1 \Rightarrow \sin x = -\frac{1}{2}$$

x Lies in (iii) or (iv) quadrant

But $\tan x < 0$

$\therefore x = -\pi/6 \quad (or)\ 11\pi/6$

But $x \in [0, 2\pi]$

$\therefore x = 11\pi/6$

16. If (i) $Tan^{-1}x + Tan^{-1}y + Tan^{-1}z = \pi$ then prove that $x + y + z = xyz$

(ii) $Tan^{-1}x + Tan^{-1}y + Tan^{-1}z = \frac{\pi}{2}$ then prove that $xy + yz + zx = 1$

Solution:

Let $\tan^{-1} x = \alpha \quad \tan^{-1} y = \beta \quad \tan^{-1} z = \gamma$

$x = Tan\alpha \quad y = Tan\beta \quad z = Tan\gamma$

Given $\alpha + \beta + \gamma = \pi \Rightarrow \alpha + \beta = \pi - x$

$Tan(\alpha + \beta) = Tan(\pi - \gamma) \Rightarrow \frac{Tan\alpha + Tan\beta}{1 - Tan\alpha Tan\beta} = Tan\gamma$

$Tan\alpha + Tan\beta = -Tan\gamma + Tan\alpha Tan\beta Tan\gamma \Rightarrow Tan\alpha + Tan\beta + Tan\gamma \; Tan\alpha Tan\beta Tan\gamma$

$= x + y + z = xyz$

17. Prove that $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{bc + ca + ab - s^2}{\Delta}$

Solution : -

$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}$

$\frac{(s-b)(s-c)}{\Delta} + \frac{(s-c)(s-a)}{\Delta} + \frac{(s-a)(s-b)}{\Delta}$

$\frac{s^2 - cs - bs + bc + s^2 - as - cs + ac + s^2 - bs - as + ab}{\Delta}$

$\frac{3s^2 - 2as - 2bs - 2cs + bc + ca + ab}{\Delta}$

$\frac{bc + ca + ab + 3s^2 - 2s(a+b+c)}{\Delta} = \frac{bc + ca + ab + 3s^2 - 2s(2s)}{\Delta}$

$\frac{bc + ca + ab + 3s^2 - 4s^2}{\Delta} = \frac{bc + ca + ab - s^2}{\Delta}$

SECTION C

18. If $f : A \to B$, $g : B \to C$ are two one one onto functions then $gof : A \to C$ is also one one be onto.

Sol: i) Let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$.

$x_1, x_2 \in A,\ f : A \to B \Rightarrow f(x_1),\ f(x_2) \in B$

$f(x_1),\ f(x_2) \in B, \to C,\ f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)] \Rightarrow (gof)(x_1) = (gof)(x_2)$

$x_1, x_2 \in A, (gof)(x_1) = (gof) : A \to C$ is one one $\Rightarrow x_1 = x_2$

$\therefore x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$\therefore f: A \to B$ is one one.

ii) Proof :let $z \in C$

$z \in C, g: B \to C$ is onto $\Rightarrow \exists y \in B \ni f(x) = y$

Now $(gof)(x) = g(y) = z$

$\therefore z \in C \Rightarrow \exists x \in A \ni (gof)(x) = z$.

$\therefore gof: A \to C$ is onto.

19. Prove that 2.3+ 3.4 + 4.5+ …. upto n terms $\dfrac{n(n^2+6n+11)}{3}$

Sol: 2, 3, 4………………… n terms $\quad t_n = 2 + (n-1)1 = n+1$

3, 4, 5……….. n terms $\quad t_n = 3 + (n-1)1 = n+2$

$$2.3 + 3.4 + 4.5 + \dots\dots + (n+1)(n+2) = \frac{n(n^2+6n+11)}{3}$$

Let S_n be the given statement

For n = 1 L.H.S = 2.3 = 6

$$R.H.S = \frac{1(1+6+11)}{3} = 6$$

L. H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\therefore 2.3 + 3.4 + 4.5 + \dots\dots + (k+1)(k+2) = \frac{k(k^2+6k+11)}{3} + (k+2)(k+3)$$

$$= \frac{k(k^2+6k+11) + 3(k^2+5k+6)}{3}$$

$$= \frac{k^3+9k^2+26k+18}{3}$$

$$= \frac{(k+1)\{k^2+8k+18\}}{3}$$

$$k = -1 \left| \begin{array}{cccc} 1 & 9 & 26 & 18 \\ 0 & -1 & -8 & -18 \\ \hline 1 & 8 & 18 & 0 \end{array} \right.$$

$$= \frac{(k+1)\{(k+1)^2 + 6(k+1) + 11\}}{3}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

20. If $|\bar{a}| = 1, |\bar{b}| = 1, |\bar{c}| = 2$ and $\bar{a} \times (\bar{a} \times \bar{c}) + \bar{b} = 0$, then find the angle between $\bar{a}$ and $\bar{c}$.

Sol. Given that $|\bar{a}| = 1, |\bar{b}| = 1, |\bar{c}| = 2$

Let $(\bar{a}, \bar{c}) = \theta$

Consider $\bar{a}\cdot\bar{c} = |\bar{a}||\bar{c}|\cos\theta$

$= (1)(2)\cos\theta$

$= 2\cos\theta \qquad \text{...(1)}$

Consider $\bar{a}\times(\bar{a}\times\bar{c})+\bar{b}=0$

$(\bar{a}\cdot\bar{c})\bar{a}-(\bar{a}\cdot\bar{a})\bar{c}+\bar{b}=\bar{0}$

$(2\cos\theta)\bar{a}-(1)\bar{c}+\bar{b}=0 \qquad \text{...(2)}$

$(2\cos\theta)\bar{a}-\bar{c}=-\bar{b}$

Squaring on both sides

$[(2\cos\theta)\bar{a}-\bar{c}]^2=(-\bar{b})^2$

$\Rightarrow (4\cos^2\theta)(\bar{a})^2+(\bar{c})^2-4\cos\theta(\bar{a}\cdot\bar{c})=\bar{b}^2$

$\Rightarrow 4\cos^2\theta(1)+(2)^2-4\cos\theta(2\cos\theta)=1$

$\Rightarrow 4\cos^2\theta+4-8\cos^2\theta=1$

$\Rightarrow 4-4\cos^2\theta=1$

$\Rightarrow 4\cos^2\theta=3$

$\Rightarrow \cos^2\theta=\frac{3}{4}\Rightarrow\cos\theta=\pm\frac{\sqrt{3}}{2}$

Case I :

If $\cos\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{6}$

$\Rightarrow(\bar{a},\bar{c})=\frac{\pi}{6}=30^\circ$

Case II :

1. If $\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}=150^\circ$

$\Rightarrow(\bar{a},\bar{c})=\frac{5\pi}{6}=150^\circ$

21. In a triangle ABC prove that

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$$

Solution:

(i) Given $A+B+C=\pi$

R.H..S

$$4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)=2\left\{\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\right\}\left\{2\cos\left(\frac{\pi-C}{4}\right)\right\}$$

$$= \left\{ \cos\left(\frac{\pi - A + \pi - B}{}\right) + \cos\left(\frac{\pi - A - \pi + B}{4}\right) \right\} \left\{ 2\cos\left(\frac{\pi - C}{4}\right) \right\}$$

$$\{\because 2\cos A\cos B = \cos(A+B) + \cos(A+B)\}$$

$$= \left\{ \cos\left\{\frac{\pi}{2} - \left(\frac{A+B}{4}\right)\right\} + \cos\left(\frac{A-B}{4}\right) \right\} 2\cos\left(\frac{\pi - C}{4}\right)$$

$$= 2\cos\frac{\pi - C}{4}\sin\left(\frac{A+B}{4}\right) + 2\cos\left(\frac{\pi - C}{4}\right)\cos\left(\frac{A-B}{4}\right) \quad \because \cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right) = \sin\left(\frac{A+B}{4}\right)$$

$$= \sin\left(\frac{\pi - C + A + B}{4}\right) - \sin\left(\frac{\pi - C - A - B}{4}\right) + \cos\left(\frac{\pi - C + A - B}{4}\right) + \cos\left(\frac{\pi - C - A + B}{4}\right)$$

$$\left\{ \begin{array}{l} \because 2\cos A\sin B = \sin(A+B) - \sin(A-B) \\ 2\cos A\cos B = \cos(A+B) + \cos(A-B) \end{array} \right\}$$

$$\therefore \sin\left(\frac{\pi - C + \pi - C}{4}\right) - \sin\left\{\frac{A + B + C - C - A - B}{4}\right\} + \cos\left(\frac{\cancel{A} + \cancel{B} + \cancel{C} - \cancel{C} + A - B}{4}\right)$$

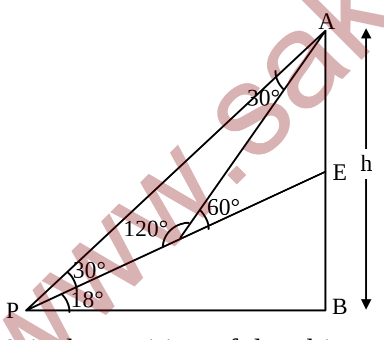
$$+ \cos\left\{\frac{\cancel{A} + \cancel{B} + \cancel{C} - \cancel{C} - \cancel{A} + \cancel{B}}{4}\right\} \left\{ \begin{array}{l} \because \pi = A + B + C \\ ad\ A + B = \pi - C \end{array} \right\}$$

$$= \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) + \cos\frac{A}{2} + \cos\frac{B}{2}$$

$$= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

22. Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

Sol.



A is the position of the object.

Given that AB = h cm

P and Q are points of observation.

Given that, PQ = 10 cm

We have,

$\angle BPE = 15°$, $\angle EPA = 30°$, $\angle EQA = 60°$

In ΔPQA,

P = 30°, Q = 120° and A = 30°

∴ By sine rule,

$$\frac{AP}{\sin 120°} = \frac{PQ}{\sin 30°}$$

$$\frac{AP}{\sin(180° - 60°)} = \frac{10}{1/2}$$

$$\frac{AP}{\sin 60°} = 20° \Rightarrow \frac{AP}{\sqrt{3}/2} = 20°$$

$$AP = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$

In $\Delta PBA,\ \sin 45° = \dfrac{AB}{AP}$

$$\frac{1}{\sqrt{2}} = \frac{h}{10\sqrt{3}}$$

$$h = \frac{10\sqrt{3}}{\sqrt{2}} = \frac{5 \cdot 2 \cdot \sqrt{3}}{\sqrt{2}} = 5\sqrt{2}\sqrt{3} = 5\sqrt{6}\text{ cm}$$

23. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. L.H.S. = $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

By applying $R_1 \Rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

By applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$= (2)(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S.}$$

24. Find the non-trivial solutions, if any, for the following equations.
$2x + 5y + 6z = 0$, $x - 3y - 8z = 0$, $3x - y - 4z = 0$

Sol. The coefficient matrix $A = \begin{bmatrix} 2 & 5 & 6 \\ 1 & -3 & -8 \\ 3 & 1 & -4 \end{bmatrix}$

On interchanging R_1 and R_2, we get

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 11 & 22 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_3 \to R_3 + 10$

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Det A = 0 as R_2 and R_3 are identical.

Clearly rank (A) = 2, as the sub-matrix $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ is non-singular.

Hence the system has non-trivial solution.

The following system of equations is equivalent to the given system of equations.

$x - 3y - 8z = 0$

$y + 2z = 0$

On giving an arbitrary value k to z, we obtain the solution set is

$x = 2k,\ y = -2k,\ z = k,\ k \in R$ for $k \neq 0$.

We obtain non-trivial solutions.