MATHEMATICS PAPER -1A

TIME: 3hrs. **Max. Marks.75**

Note: This question paper consists of three sections A, B and C.

SECTION A

Very Short Answer Type Questions. **10 X 2 = 20**

1. Find the domain of the function $f(x) = \dfrac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$.

2. If f(x + y) = f(xy) $\forall$x, y $\in$ R then prove that f is a constant function.

3. ABCDE is a pentagon. If the sum of the vectors AB, AE, BC, DE, ED and AC is λAC, then find the value of λ.

4. If the position vectors of the points A, B, C are $-2\bar{i} + \bar{j} - \bar{k}, -4\bar{i} + 2\bar{j} + 2\bar{k}$ and $6\bar{i} - 3\bar{j} - 13\bar{k}$ respectively and $\overline{AB} = \lambda\overline{AC}$ then find the value of λ.

5. Find the area of the triangle having $3\bar{i} + 4\bar{j}$ and $-5\bar{i} + 7\bar{j}$ as two of its sides.

6. Find the value of $\sin 330^0 . \cos 120^0 + \cos 210^0 . \sin 300^0$.

7. Find the value of $\cos^2 52\frac{1}{2}° - \sin^2 22\frac{1}{2}°$.

8. Prove that $\tan h3x = \dfrac{3\tan hx + \tanh^3 x}{1 + 3\tan h^2 x}$.

9. If I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and E = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.

10. If A = $\begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = 0$, then find the value of k.

SECTION B

Short answer type questions.
Answer any five of the following. **5 X 4 = 20**

11. If $\theta - \phi = \frac{\pi}{2}$, show that $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = 0$

12. If a, b, c are non-coplanar vectors, then test for the collinearity of the following points whose position vectors are given.

13. Find a unit vector perpendicular to the plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1).

14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}$.

15. If x is acute and $\sin\left(x+10^0\right) = \cos\left(3x - 68^0\right)$ then find x

16. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then prove that
$x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2\left\{x^2y^2 + y^2z^2 + z^2x^2\right\}$

17. In a triangle ABC if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ then show that $c = 60^0$

SECTION C

Long answer type questions.
Answer any five of the following. **5 X 7 = 35**

18. Show that $49^n + 16n - 1$ is divisible by 64 for all positive integers n.

19. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5) find the shortest distance between AB and CD.

20. If A, B, C are the angles of a triangle then prove that $\cos 2A + \cos 2B + \cos 2C = -4\cos A\cos B\cos C - 1$

21. In any triangle with usual notation, If $r_1 + r_2 + r_3 = r$, then show that $\angle C = 90°$.

22. Solve the equations $2x - y + 8z = 13$, $3x + 4y + 5z = 18$, $5x - 2y + 7z = 20$ by Matrix inversion method

23. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

24. If $f : A \to B$ is a bijection then show that $f^{-1}of = I_A$ and $fof^{-1} = I_B$.

Modal Paper – 1 Solution

1. Find the domain of the function $f(x)=\dfrac{2x^2-5x+7}{(x-1)(x-2)(x-3)}$.

Sol. $f(x)=\dfrac{2x^2-5x+7}{(x-1)(x-2)(x-3)}$

$(x-1)(x-2)(x-3)\neq 0$

$\Rightarrow x-1\neq 0, x-2\neq 0, x-3\neq 0$

$\Rightarrow x\neq 1, x\neq 2, x\neq 3$

$\Rightarrow x\in R-\{1,2,3\}$

$\therefore$ Domain of f is $R-\{1,2,3\}$

2. If f(x + y) = f(xy) $\forall x, y\in R$ then prove that f is a constant function.

Sol. f(x + y) = f(xy)

Let f(0) = k

then f(x) = f(x + 0) = f(x· 0) = f(0) = k

$\Rightarrow$f(x + y) = k

$\therefore$ f is a constant function.

3. ABCDE is a pentagon. If the sum of the vectors AB, AE, BC, DE, ED and AC is λAC, then find the value of λ.

Sol. Given that,

$\overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED}+\overline{AC}=\lambda\overline{AC}$

$\Rightarrow(\overline{AB}+\overline{BC})+(\overline{AE}+\overline{ED})+(\overline{DC}+\overline{AC})=\lambda\overline{AC}$

$\Rightarrow\overline{AC}+\overline{AD}+\overline{DC}+\overline{AC}=\lambda\overline{AC}$

$\Rightarrow\overline{AC}+\overline{AC}+\overline{AC}=\lambda\overline{AC}$

$\Rightarrow 3\overline{AC}=\lambda\overline{AC}$

$\therefore\lambda=3$

4. If the position vectors of the points A, B, C are $-2\bar{i}+\bar{j}-\bar{k}, -4\bar{i}+2\bar{j}+2\bar{k}$ and $6\bar{i}-3\bar{j}-13\bar{k}$ respectively and $\overline{AB}=\lambda\overline{AC}$ then find the value of λ.

Sol. Let O be the origin and $\overline{OA}=-2\bar{i}+\bar{j}-\bar{k}$, $\overline{OB}=-4\bar{i}+2\bar{j}+2\bar{k}$, $\overline{OC}=6\bar{i}-3\bar{j}-13\bar{k}$

Given $\overline{AB}=\lambda\overline{AC}$

$$\overline{OB}-\overline{OA}=\lambda\left[\overline{OC}-\overline{OA}\right]$$

$$-4\bar{i}+2\bar{j}+2\bar{k}+2\bar{i}-\bar{j}+\bar{k}=$$

$$\lambda\left[6\bar{i}-3\bar{j}-13\bar{k}+2\bar{i}-\bar{j}+\bar{k}\right]$$

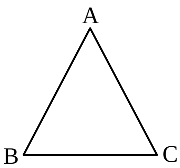
$$-2\bar{i}+\bar{j}+3\bar{k}=\lambda\left[8\bar{i}-\bar{j}-12\bar{k}\right]$$

Comparing $\bar{i}$ coefficient on both sides

$$-2=\lambda 8 \Rightarrow \lambda=-\frac{2}{8}\Rightarrow \lambda=\frac{-1}{4}$$

5. Find the area of the triangle having $3\bar{i}+4\bar{j}$ and $-5\bar{i}+7\bar{j}$ as two of its sides.

Sol.



Given $\overline{AB}=3\bar{i}+4\bar{j},\ \overline{BC}-5\bar{i}+7\bar{j}$

We know that,

$$\overline{AB}+\overline{BC}+\overline{CA}=0$$

$$\overline{CA}=-\overline{AB}-\overline{BC}=-3\bar{i}-4\bar{j}+5\bar{i}-7\bar{j}$$

$$\overline{CA}=2\bar{i}-11\bar{j}$$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}|\overline{AB}\times\overline{AC}|$

$$=\frac{1}{2}\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 4 & 0\\ 2 & -11 & 0\end{vmatrix}=\frac{1}{2}\left[\bar{k}(-33-8)\right]$$

$$=\left|\frac{-41\bar{k}}{2}\right|=\frac{41}{2}$$

6. Find the value of $\sin 330^0 . \cos 120^0+\cos 210^0 . \sin 300^0$

Solution:

$$\sin 330^0=\sin\left(360^0-30^0\right)=-\sin 30^0=-\frac{1}{2}$$

$$\cos 120^0=-\frac{1}{2}\cos 210^0=-\frac{\sqrt{3}}{2};\sin 300^0=\sin\left(360^0-60^0\right)=-\frac{\sqrt{3}}{2}$$

$$\sin 330^0 \cos 120^0 + \cos 210^0 \,.\sin 300^0 = -\frac{1}{2}\times -\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = 1$$

7. Find the value of $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

Sol. $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

$$[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$$

$$= \cos\left(52\frac{1}{2}^\circ + 22\frac{1}{2}^\circ\right)\cos\left(52\frac{1}{2}^\circ - 22\frac{1}{2}^\circ\right)$$

$$= \cos 75^\circ \cos 30^\circ$$

$$= \frac{\sqrt{3}}{2}(\cos 75^\circ)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{3-\sqrt{3}}{4\sqrt{2}}$$

8. Prove that $\tan h3x = \dfrac{3\tan hx + \tanh^3 x}{1+3\tan h^2 x}$

Sol.

$$\tanh 3x = \tanh(2x+x) = \frac{\tan hx + \tan h2x}{1+\tan hx \tan h2x}$$

$$= \frac{\tan hx + \dfrac{2\tan hx}{1+\tanh^2 x}}{1+\tan hx \dfrac{2\tan hx}{1+\tanh^2 x}}$$

$$= \frac{\tan hx\left(1+\tanh^2 x\right) + 2\tan hx}{1+\tanh^2 x + \tan hx.2\tan hx}$$

$$= \frac{\tanh^3 x + 3\tan hx}{1+\tanh^2 x + 2\tan h^2 x} = \frac{\tanh^3 x + 3\tan hx}{1+3\tan h^2 x}$$

9. If I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and E = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.

Sol. $aI + bE = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$$(aI + bE)^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$(aI + bE)^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$$

$$= a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= a^3I + 3a^2bE$$

10. If $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = 0$, then find the value of k.

Sol. $A^2 = 0 \Rightarrow \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 4-4 & 8+4k \\ -2-k & -4+k^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow 8 + 4k = 0 \Rightarrow 4k = -8 \Rightarrow k = -2$

SECTION B

11. If $\theta - \phi = \dfrac{\pi}{2}$, show that $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = 0$

Sol. Given $\theta - \phi = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{2} + \phi$

$$\cos\theta = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$\sin\theta = \sin\left(\frac{\pi}{2} + \phi\right) = \cos\phi$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix}$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix}\sin^2\phi & -\sin\phi\cos\phi\\ -\sin\phi\cos\phi & \cos^2\phi\end{bmatrix}$$

$$\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\sin^2\phi\cos^2\phi-\sin^2\phi\cos^2\phi & \sin^3\phi\cos\phi-\sin^3\phi\cos\phi\\ -\sin\phi\cos^3\phi+\sin\phi\cos^3\phi & -\sin^2\phi\cos^2\phi+\sin^2\phi\cos^2\phi\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

12. If a, b, c are non-coplanar vectors, then test for the collinearity of the following points whose position vectors are given.

i) Show that $\bar{a}-2\bar{b}+3\bar{c}, 2\bar{a}+3\bar{b}-4\bar{c}$, $-7\bar{b}+10\bar{c}$ are collinear.

Sol. Let $\overline{OA}=\bar{a}-2\bar{b}+3\bar{c}$, $\overline{OB}=2\bar{a}+3\bar{b}-4\bar{c}$

$\overline{OC}=-7\bar{b}+10\bar{c}$

$\overline{AB}=\overline{OB}-\overline{OA}=\bar{a}+5\bar{b}-7\bar{c}$

$\overline{AC}=\overline{OC}-\overline{OA}=-\bar{a}-5\bar{b}+7\bar{c}$

$\overline{AC}=-\bar{a}-5\bar{b}+7\bar{c}=-[\bar{a}+5\bar{b}-7\bar{c}]$

$\overline{AC}=-\overline{AB}$

$\overline{AC}=\lambda\overline{AB}$ where $\lambda=-1$

$\therefore$ Given vectors are collinear.

13. Find a unit vector perpendicular to the plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1).

Sol. Let O be the origin and

$\overline{OP}=\bar{i}-\bar{j}+2\bar{k}, \overline{OQ}=2\bar{i}-\bar{k}, \overline{OR}=2\bar{j}+\bar{k}$

$\overline{PQ} = \overline{OQ} - \overline{OP} = \overline{i} - 2\overline{k}$

$\overline{PR} = \overline{OR} - \overline{OP} = -\overline{i} + 3\overline{j} - \overline{k}$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & 0 & -2 \\ -1 & 3 & -1 \end{vmatrix}$$

$= \overline{i}(0+6) - \overline{j}(-1-2) + \overline{k}(3-0)$

$\overline{PQ} \times \overline{PR} = 6\overline{i} + 3\overline{j} + 3\overline{k}$

$|\overline{PQ} \times \overline{PR}| = 3\sqrt{4+1+1} = 3\sqrt{6}$

$\therefore$ The unit vector perpendicular to the plane passing through

P, Q and R is $= \pm \dfrac{\overline{PQ} \times \overline{PR}}{|\overline{PQ} \times \overline{PR}|}$

$$= \pm \frac{3(2\overline{i} + \overline{j} + \overline{k})}{3\sqrt{6}} = \pm \frac{2\overline{i} + \overline{j} + \overline{k}}{\sqrt{6}}$$

14. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}$.

Sol. L.H.S. =

$$\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left[1+\cos\left(\pi-\frac{3\pi}{10}\right)\right]\left[1+\cos\left(\pi-\frac{\pi}{10}\right)\right]$$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$= \left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right)$$

$$= \sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10} = \left[\sin\frac{\pi}{10}\right]^2\left[\sin\frac{3\pi}{10}\right]^2$$

$$= \sin^2 18^\circ \sin^2 54^\circ$$

$$= \left[\frac{\sqrt{5}-1}{4}\right]^2\left[\frac{\sqrt{5}+1}{4}\right]^2 = \frac{(\sqrt{5}-1)^2}{16} \times \frac{(\sqrt{5}+1)^2}{16}$$

$$= \frac{[(\sqrt{5}-1)(\sqrt{5}+1)]^2}{16\times16} = \frac{(5-1)^2}{16\times16} = \frac{4^2}{16\times16} = \frac{16}{16\times16} = \frac{1}{16}$$

15. If x is acute and $\sin\left(x+10^0\right) = \cos\left(3x-68^0\right)$ then find x

Sol. Given $\sin\left(x+10^0\right) = \cos\left(3x-68^0\right)$

$$\Rightarrow \sin(x+10) = \sin\left(90^0 + 3x - 68\right)$$

$$= \sin\left(22^0 + 3x\right)$$

$$\therefore x + 10 = n\pi + (-1)^n\left(22^0 + 3x\right)$$

If n is even, then

$$x + 10 = n\pi + \left(22^0 + 3x\right)$$

$\Rightarrow$ x = $\dfrac{n\pi\text{-}12}{2}$ *which* is not acute

If n is odd, then

$$x + 10 = (2k+1)\pi - \left(22^0 + 3x\right), n = 2k+1$$

$\Rightarrow$ x = $(2k+1)\dfrac{\pi}{4}$ -8^0

If K=0, then x=37^0.

16. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then prove that $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2\left\{x^2y^2 + y^2z^2 + z^2x^2\right\}$

Let $\sin^{-1}x = \alpha \quad \sin^{-1}y = \beta \quad \sin^{-1}z = \gamma$

$\sin\alpha = x \quad \sin\beta = y \quad \sin\gamma = z$

$$\alpha + \beta + \gamma = \pi \Rightarrow \cos(\alpha + \beta) = \cos(\pi - \gamma)$$

$$\sqrt{1-x^2}\sqrt{1-y^2} - xy = -\sqrt{1-z^2}$$

$$\sqrt{1-x^2}\sqrt{1-y^2} = xy - \sqrt{1-z^2}$$

$$(1-x)^2(1-y)^2 = x^2y^2 + 1 - z^2 - 2xy\sqrt{1-z^2}$$

$$1 - x^2 - y^2 + x^2y^2 = x^2y^2 + 1 - z^2 - 2xy\sqrt{1-z^2}$$

Squaring on both sides we have

$$z^2 - x^2 - y^2 = -2xy\sqrt{1-z^2}$$

$$z^4 + x^4 + y^4 - 2x^2z^2 + 2x^2y^2 - 2y^2z^2 = 4x^2y^2(1-z)^2$$

$$x^4 + y^4 + z^4 - 4x^2y^2z^2 = 2x^2y^2 + 2y^2z^2 + 2x^2z^2$$

$\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \alpha$ then prove that $\dfrac{x^2}{az} - \dfrac{2xy}{ab}\cos\alpha + \dfrac{y^2}{b^2} = \sin^2\alpha$

Let $\cos^{-1}\dfrac{x}{a} = \theta$ *and* $\cos^{-1}\dfrac{y}{b} = \phi$

$\cos\theta = \frac{x}{a} \quad \cos\phi = \frac{y}{b} = \theta$

Given $\theta + \phi = \alpha$

$\cos(\theta + \phi) = \cos\alpha \Rightarrow \cos\theta\cos\phi - \sin\theta\sin\phi$

$\frac{xy}{ab} - \sqrt{1 - \frac{x^2}{a^2}}\ \sqrt{1 - \frac{y^2}{b^2}} = \cos\alpha$

$\frac{xy}{ab} - \cos\alpha = \sqrt{1 - \frac{x^2}{a^2}}\ \sqrt{1 - \frac{y^2}{b^2}}$

Squaring on both sides we have

$\frac{x^2y^2}{a^2b^2} + \cos^2\alpha = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)$

$\frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\cos\alpha$

$\frac{x^2y^2}{a^2b^2} + \cos^2\alpha = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2y^2}$

$\frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\cos\alpha$

$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = 1 - \cos^2\alpha$

$\therefore \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = 1 - \sin^2\alpha$

17. In a triangle ABC $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ show that $c = 60^0$

Solution : -

Given $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$\frac{a+b+c}{a+c} + \frac{a+b+c}{b+c} = 3$

$\frac{(a+c)+b}{a+c} + \frac{a+(b+c)}{b+c} = 3 \Rightarrow \frac{\cancel{a}+c}{\cancel{a}-c} + \frac{b}{a+c} + \frac{a}{b+c} + \frac{\cancel{b+c}}{\cancel{b+c}} = 3$

$\frac{b}{a+c} + \frac{a}{b+c} = 1 \Rightarrow \frac{b^2+bc+a^2+ac}{ab+ac+bc+c^2} = 1$

$a^2 + b^2 - c^2 = ab \Rightarrow 2ab\cos C = ab \left\{\because a^2 + b^2 - c^2 = 2abc\cos C\right\}$

$$\cos C = \frac{ab}{2ab} = \frac{1}{2} \Rightarrow C = 60^0 \quad \cos C = \frac{\cancel{ab}}{\cancel{2ab}} = \frac{1}{2} \Rightarrow C = 60^0$$

SECTION C

18. Show that $49^n + 16n - 1$ is divisible by 64 for all positive integers n.

Sol. Let S(n) be the statement.

$49^n + 16n - 1$ is divisible by 64.

Since $49^1 + 16.1 - 1 = 64$ is divisible by 64.

$\therefore$ S(n) is true for n = 1

Assume that the statement S(n) is true for n = k

i.e. $49^n + 16n - 1$ is divisible by 64

Then $49^k + 16k - 1 = 64\,M$ …(1)

($\because$ M is an integer)

We show that the statement S(n) is true for n = k + 1

i.e. we show that $49^{k+1} + 16(k+1) - 1$ is divisible by 64.

From (1), we have

$49^k + 16k - 1 = 64\,M$

$49^k = 64\,M - 16k + 1$

$49^k \times 49 = (64\,M - 16k + 1) \times 49$

$49^{k+1} + 16(k + 1) - 1 = (64M - 16k + 1)49 + 16(k + 1) - 1$

$= 64 \times 49\,M - 49 \times 16k + 49 + 16k + 16 - 1$

$= 64 \times 49\,M - 48 \times 16k + 64$

$= 64 \times 49\,M - 64 \times 12k + 64$

$= 64(49\,M - 12k + 1)$

$= 64\,N$ [$\because$ N is an integer]

$\therefore$ S(n) is true for n = k + 1

$\therefore$ By the principle of mathematical induction, S(n) is true for all $n \in N$.

19. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5) find the shortest distance between AB and CD.

Sol. Let O be the origin

Let $\overline{OA} = \overline{i} - 2\overline{j} - \overline{k}, \overline{OB} = 4\overline{i} - 3\overline{k}$

$\overline{OC} = \overline{i} + 2\overline{j} - \overline{k}, \overline{OD} = 2\overline{i} - 4\overline{j} - 5\overline{k}$

The vector equation of a line passing through A, B is

$$\bar{r} = (1-t)\bar{a} + t\bar{b}, t \in R$$

$$= \bar{a} + t(\bar{b} - \bar{a})$$

$$= \bar{i} - 2\bar{j} - \bar{k} + t(4\bar{i} - 3\bar{k} - \bar{i} + 2\bar{j} + \bar{k})$$

$$= \bar{i} - 2\bar{j} - \bar{k} + t(3\bar{i} + 2\bar{j} - 2\bar{k})$$

$$= \bar{a} + t\bar{b}$$

where $\bar{a} = \bar{i} - 2\bar{j} - \bar{k}, \bar{b} = 3\bar{i} + 2\bar{j} - 2\bar{k}$

The vector equation of a line passing through C, D is

$$\bar{r} = (1-s)\bar{c} + s\bar{d}, s \in R$$

$$\bar{r} = \bar{c} + s(\bar{d} - \bar{c})$$

$$= \bar{i} + 2\bar{i} - \bar{k} + s[2\bar{i} - 4\bar{j} - 5\bar{k} - \bar{i} - 2\bar{j} + \bar{k}]$$

$$= \bar{i} + 2\bar{j} - \bar{k} + s[\bar{i} - 6\bar{j} - 4\bar{k}]$$

$$= \bar{c} + s\bar{d}$$

where $\bar{c} = \bar{i} + 2\bar{j} - \bar{k}, \bar{d} = \bar{i} - 6\bar{j} - 4\bar{k}$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix}$$

$$= \bar{i}[-8-12] - \bar{j}[-12+2] + \bar{k}[-18-2]$$

$$= -20\bar{i} + 10\bar{j} - 20\bar{k} = 10[-2\bar{i} + \bar{j} - 2\bar{k}]$$

$$|\bar{b} \times \bar{d}| = 10\sqrt{4+1+4} = 10 \cdot 3 = 30$$

$$\bar{a} - \bar{c} = \bar{i} - 2\bar{j} - \bar{k} - \bar{i} - 2\bar{j} + \bar{k} = -4\bar{j}$$

$$\frac{[\bar{a} - \bar{c}\ \ \bar{b} \cdot \bar{d}]}{|\bar{b} \times \bar{d}|} = \frac{(\bar{a} - \bar{c}) \cdot (\bar{b} \times \bar{d})}{|\bar{b} \times \bar{d}|}$$

$$= \frac{-4\bar{j} \cdot 10[-2\bar{i} + \bar{j} - 2\bar{k}]}{30} = \frac{10[4]}{30} = \frac{40}{30} = \frac{4}{3}$$

$\therefore$ The shortest distance between the lines = 4/3.

20. . If A, B, C are the angles of a triangle then prove that cos2A+cos2B+cos2C = -4cosAcosBcosC-1

SOL. cos2A+cos2B+cos2C =

$= 2\cos\frac{2A+2B}{2}\cos\frac{2A-2B}{2} + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$

$= 2\cos(\pi - c)\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$

$= 2\cos C(-\cos(A-B) + \cos C) - 1$

$= 2\cos C(-\cos(A-B) + \cos(\pi - (A+B))) - 1$

$= 2\cos C(-\cos(A-B) - \cos(A+B)) - 1$

$= 2\cos C(-2\cos A\cos B) - 1$

$= -4\cos A\cos B\cos C - 1$

21. In any triangle with usual notation , If $r_1 + r_2 + r_3 = r$, then show that $\angle C = 90°$.

Sol. $r_1 + r_2 = r - r_3 \Rightarrow \frac{r_1 + r_2}{r - r_3} = 1 \qquad \ldots(1)$

$r_1 + r = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$= 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right]$

$= 4R\cos\frac{C}{2}\left[\sin\frac{A+B}{2}\right]$

$= 4R\cos\frac{C}{2}\cdot\cos\frac{C}{2}$

$= 4R\cos^2\frac{C}{2}$

$$r - r_3 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} -$$

$$4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2} - \cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= 4R\sin\frac{C}{2}\left[-\cos\left(\frac{A+B}{2}\right)\right]$$

$$= 4R\sin\frac{C}{2}\left[-\sin\frac{C}{2}\right]$$

$$= -4R\sin^2\frac{C}{2}$$

$$\frac{r - r_3}{r_1 + r_2} = \frac{4R\sin^2\frac{C}{2}}{4R\cos^2\frac{C}{2}} = \tan^2\frac{C}{2}$$

$$\therefore \tan^2\frac{C}{2} = \tan 45° \qquad \text{From(1)}$$

$$\frac{C}{2} = 45° \qquad \therefore \angle C = 90°$$

22. solve the equations $2x - y + 8z = 13$
$3x + 4y + 5z = 18$
$5x - 2y + 7z = 20$ by Matrix inversion method

Sol. Matrix inversion method :

Let $A = \begin{bmatrix} 2 & -1 & 8 \\ 3 & 4 & 5 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = 28 + 10 = 38$$

$$B_1 = -\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -(21 - 25) = 4$$

$$C_1 = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$$

$$A_2 = -\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = -(-7+16) = -9$$

$$B_2 = \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = (14-40) = -26$$

$$C_2 = -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -(-4+5) = -1$$

$$A_3 = \begin{vmatrix} -1 & 8 \\ 4 & 5 \end{vmatrix} = -5-32 = -37$$

$$B_3 = -\begin{vmatrix} 2 & 8 \\ 3 & 5 \end{vmatrix} = -(10-24) = 14$$

$$C_3 = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8+3 = 11$$

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$\begin{aligned} \text{Det}\,A &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= 2\cdot 38 + (-1)4 + 8(-26) \\ &= 76 - 4 - 208 = -136 \end{aligned}$$

$$A^{-1} = \frac{\text{Adj}\,A}{\text{Det}\,A} = -\frac{1}{136}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{280}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}\begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} 494 & -162 & -740 \\ 52 & -468 & +280 \\ -338 & -18 & +220 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} -408 \\ -136 \\ -136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Solution is x = 3, y = 1, z = 1.

23. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Sol. L.H.S. $C_1 \to C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$(R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3 = \text{R.H.S}$$

24. **If $f : A \to B$ is a bijection then show that $f^{-1}of = I_A$ and $fof^{-1} = I_B$.**

Proof :

Since $f : A \to B$ is bijection we have its inverse $f^{-1} : B \to A$ is also a bijection.

$f : A \to B$ and $f^{-1} : B \to A$ then $f^{-1}of : A \to A$

$f^{-1}of$ and I_A have same domain A.

Let $a \in A$, since $f^{-1} : B \to A$ is onto, there exists $b \in B$ such that $f^{-1}(b) = a \Rightarrow f(a) = b$.

$(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(b) = a = I_A(a)$

$\therefore f^{-1}of = I_A$.

$f^{-1}: B \to A$ and $f : A \to B \Rightarrow fof^{-1} : B \to B$.

fof^{-1} and I_B have same domain B.

Let $b \in B$, since $f : A \to B$ is onto, there exists $a \in A$ such that $f(a) = b f^{-1}(b) = a$.

$(fof^{-1})(b) = f(f^{-1}(b)) = f(a) = b = I_B(b)$.

$\therefore fof^{-1} = I_B$