MATHEMATICAL METHODS

# REAL AND COMPLEX MATRICES QUATRADTIC FORMS

I YEAR B.Tech

**By**

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# SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

| Name of the Unit | Name of the Topic |
|---|---|
| **Unit-I**<br>**Solution of Linear systems** | **Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems. |
| **Unit-II**<br>**Eigen values and Eigen vectors** | Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices. |
| **Unit-III**<br>**Linear Transformations** | Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition. |
| **Unit-IV**<br>**Solution of Non-linear Systems** | **Solution of Algebraic and Transcendental Equations**- Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:**Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline. |
| **Unit-V**<br>**Curve fitting & Numerical Integration** | **Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares.<br>**Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature. |
| **Unit-VI**<br>**Numerical solution of ODE** | Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method. |
| **Unit-VII**<br>**Fourier Series** | Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions. |
| **Unit-VIII**<br>**Partial Differential Equations** | Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation. |

# CONTENTS

UNIT-III

## REAL AND COMPLEX MATRICES, QUADRATIC FORMS

- **Definitions of Hermitian and skew Hermitian matrices**
- **Quadratic forms**
- **Types of Quadratic foms**
- **Canonical form**

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- **Conjugate Matrix:** Suppose $A$ is any matrix, then the conjugate of the matrix $A$ is denoted by $\bar{A}$ and is defined as the matrix obtained by taking the conjugate of every element of $A$.
  - Conjugate of $a + ib$ is $a - ib$
  - $\overline{(\bar{A})} = A$
  - $\overline{A.B} = \bar{A}.\bar{B}$
  - $\overline{A+B} = \bar{A} + \bar{B}$

  Ex: If $A = \begin{bmatrix} 1 & 2+3i \\ 3-4i & -2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1 & 2-3i \\ 3+4i & 2i \end{bmatrix}$

- **Conjugate Transpose of a matrix (or) Transpose conjugate of a matrix:** Suppose $A$ is any square matrix, then the transpose of the conjugate of $A$ is called Transpose conjugate of $A$. It is denoted by $A^{\theta} = (\bar{A})^T = \overline{(A^T)}$.

  Ex: If $A = \begin{bmatrix} 1-i & -2i \\ 4-3i & 5-4i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 1+i & 2i \\ 4+3i & 5+4i \end{bmatrix}$

  Now, $(\bar{A})^T = \begin{bmatrix} 1+i & 4+3i \\ 2i & 5+4i \end{bmatrix} = A^{\theta}$
  - $\left(A^{\theta}\right)^{\theta} = A$
  - $(A+B)^{\theta} = A^{\theta} + B^{\theta}$
  - $(AB)^{\theta} = B^{\theta} A^{\theta}$

- **Hermitian Matrix:** A square matrix $A$ is said to be Hermition if $A^{\theta} = A$

  Ex: If $A = \begin{bmatrix} 2 & 5+i \\ 5-i & 6 \end{bmatrix}$ is a Hermition matrix
  - The diagonal elements of Hermitian matrix are purely Real numbers.
  - $A$ is Hermition $\Rightarrow a_{ij} = \begin{cases} real\ if & i = j \\ \overline{a_{ji}}\ \ if & i \neq j \end{cases}$
  - The number of Independent elements in a Hermitian matrix are $\frac{n(n+1)}{2}$, $n$ is Order.

- **Skew Hermitian Matrix:** A square matrix $A$ is said to be Skew Hermition if $A^{\theta} = -A$

  Ex: If $A = \begin{bmatrix} i & 3+i & 4 \\ -3+i & 0 & 6 \\ -4 & -6 & 3i \end{bmatrix}$ is a Skew Hermition Matrix.
  - The diagonal elements of Skew Hermition matrix are either '0' or Purely Imaginary.
  - $A$ is Skew Hermition $\Rightarrow a_{ij} = \begin{cases} Imaginary\ (or)\ 0 & if\ \ i = j \\ \overline{a_{ji}} & if\ \ i \neq j \end{cases}$
  - The no. of Independent elements in a Skew Hermitian matrix are $\frac{n(n-1)}{2}$ , $n$ is Order

- **Orthogonal Matrix:** A square matrix $A$ is said to be Orthogonal if $A\,A^T = A^T A = I$

  Ex: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

- If $A$ is orthogonal, then $A^T$ is also orthogonal.
- If $A, B$ are orthogonal matrices, then $AB$ is orthogonal.

- Unitary Matrix: A square matrix $A$ is said to be Unitary matrix if $A\,A^{\theta} = A^{\theta}A = I$
  - If $A$ is a Unitary matrix, then $A^T, A^{\theta}$ are also Unitary.
  - If $A, B$ are Unitary matrices, then $AB$ is Unitary.
- Normal Matrix: A square matrix $A$ is said to be Normal matrix if
  - i. $A\,A^T = A^T A$ (if $A$ is Real)
  - ii. $A\,A^{\theta} = A^{\theta} A$ (if $A$ is non-real i.e. Complex)
  - Orthogonal and Unitary matrices are Normal Matrices.
  - Symmetric and Hermition matrices are Normal Matrices.

# Quadratic Forms

Definition: An expression of the form $Q = X^T AX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$ , where $a_{ij}$'s are constants, is called a quadratic form in $n$ variables $x_1, x_2, \dots, x_n$.

If the constants $a_{ij}$'s are real numbers, it is called a real quadratic form.

The second order homogeneous expression in $n$ variables is called a Quadratic form.

Examples

1) $3x^2 + 5xy + 3y^2$ is a quadratic form in 2 variables $x$ and $y$.
2) $3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ is a quadratic form in 3 variables $x_1, x_2, x_3$ etc.

Canonical Form: The Quadratic form which is in the form of sum of squares.

Let $X^T AX$ be a Quadratic form.

Let $X = PY$ be the transformation used for transforming the quadratic form to canonical form.

i.e. $X^T AX = (PY)^T A(PY)$

$= Y^T P^T A\,P\,Y$

This is the canonical form when $P^T A\,P = D$, where $D$ is a diagonal matrix.

There are two types of Transformations:

- Orthogonal Transformation (in which $P$ is Orthogonal)
- Congruent Transformation (in which $P$ is non-singular matrix)

Index of a Real Quadratic Form

When the quadratic form $X^T A\,X$ is reduced to the canonical form, it will contain only $r$ terms, if the rank of $A$ is $r$. The terms in the canonical form may be positive, zero or negative.

The number of positive terms in a normal form of quadratic form is called the index (s) of the quadratic form. It is denoted by s.

- The number of positive terms in any two normal reductions of quadratic form is the same.
- The number of negative terms in any two normal reductions of quadratic form is the same.

### Signature of a Quadratic Form

If $r$ is the rank of a quadratic form and $s$ is the number of positive terms in its normal form, then $(2s - r)$ will give the signature of the quadratic form.

## Types of Quadratic Forms (or) Nature of Quadratic Forms

There are five types of Quadratic Forms

- Positive definite
- Negative definite
- Positive semi definite
- Negative semi definite
- Indefinite

The Quadratic form $X^T A X$ in $n$ variables is said to be

- Positive definite: All the Eigen values of $A$ are positive.
- Negative definite: All the Eigen values of $A$ are negative.
- Positive semi definite: All the Eigen values of $A$ are $\geq 0$, and atleast one eigen value is zero.
- Negative semi definite: All the Eigen values of $A$ are $\leq 0$, and atleast one eigen value is zero.
- Indefinite: All the Eigen values of $A$ has positive as well as Negative Eigen values.

### Procedure to Reduce Quadratic form to Canonical form by Orthogonal Transformation

Step 1: Write the coefficient matrix $A$ associated with the given quadratic form.

Step 2: Find the eigen values of $A$.

Step 3: Write the canonical form using $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ \ldots + \lambda_n y_n^2$.

Step 4: Form a matrix $P$ containing the normalized eigen vectors of $A$. Then $X = PY$ gives the required orthogonal transformation, which reduces Quadratic form to canonical form.

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