MATHEMATICAL METHODS

NUMERICAL SOLUTIONS OF ALGEBRAIC AND TRANSDENTIAL EQUATIONS

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I **Solution of Linear systems**	**Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II **Eigen values and Eigen vectors**	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III **Linear Transformations**	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV **Solution of Non-linear Systems**	**Solution of Algebraic and Transcendental Equations-** Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:** Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V **Curve fitting & Numerical Integration**	**Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. **Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI **Numerical solution of ODE**	Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII **Fourier Series**	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII **Partial Differential Equations**	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-IV (a)

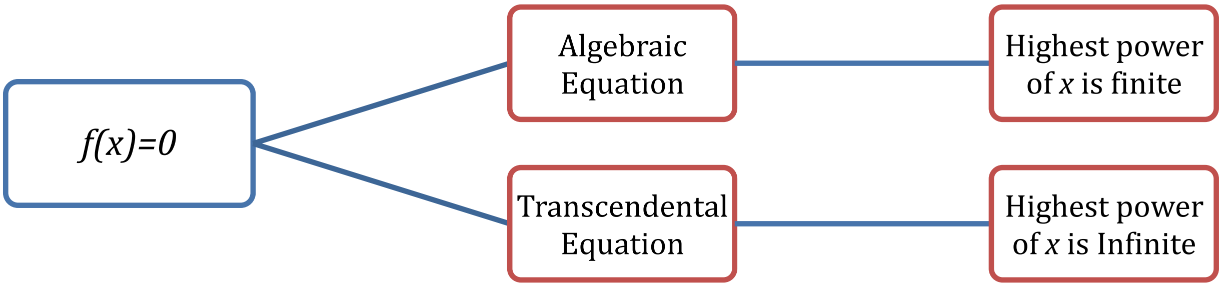
SOLUTIONS OF NON-LINEAR SYSTEMS

a) Numerical Solutions of Algebraic and Transcendental Equation

- **Introduction**
- **Bisection Method**
- **Regular Folsi Method**
- **Newton Raphson Method**
- **Iteration Method**

NUMERICAL SOLUTIONS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Aim: To find a root of $f(x) = 0$



Algebraic Equation: An Equation which contains algebraic terms is called as an algebraic Equation.

Example: $x^2 + x + 1 = 0$, Here Highest power of x is finite. So it is an algebraic Equation.

Transcendental Equation: An equation which contains trigonometric ratios, exponential function and logarithmic functions is called as a Transcendental Equation.

Example: $e^x + 2 = 0, \sin x + 1 = 0, \log(1 + x) = 0$ etc.

In order to solve above type of equations following methods exist

- **Directive Methods:** The methods which are used to find solutions of given equations in the direct process is called as directive methods.

  Example: Synthetic division, remainder theorem, Factorization method etc

  Note: By using Directive Methods, it is possible to find exact solutions of the given equation.

- **Iterative Methods (Indirect Methods):** The methods which are used to find solutions of the given equation in some indirect process is called as Iterative Methods

  Note: By using Iterative methods, it is possible to find approximate solution of the given equation and also it is possible to find single solution of the given equation at the same time.

To find a root of the given equation, we have following methods

- Bisection Method
- The Method of false position (Or) Regular folsi Method
- Iteration Method (Successive approximation Method)
- Newton Raphson Method.

Bisection Method

Consider $f(x) = 0$ be the given equation.

Let us choose the constants a and b in such a way that $f(a).f(b) < 0$

(I.e. $f(a)$ and $f(b)$ are of opposite signs) i.e. $f(a) < 0, f(b) > 0$

$$(or) \qquad f(a) > 0, f(b) < 0$$

Then the curve $y = f(x)$ crosses $x\text{-}axis$ in between a and b, so that there exists a root of the given equation in between a and b.

Let us define initial approximation $x_0 = \frac{a+b}{2}$

Now, If $f(x_0) = 0$ then x_0 is root of the given equation.

If $f(x_0) \neq 0$ then either $f(x_0) < 0$ (or) $f(x_0) > 0$

Case (i): Let us consider that $f(x_0) < 0$ and $f(a) > 0$ Since $f(x_0)\, f(a) < 0$ $\Rightarrow$ Root lies between x_0 and a $\Rightarrow x_1 = \frac{x_0+a}{2}$	Case (ii): Let us consider that $f(x_0) > 0$ and $f(b) < 0$ Since $f(x_0)\, f(b) < 0$ $\Rightarrow$ Root lies between x_0 and a $\Rightarrow x_1 = \frac{x_0+b}{2}$

Here $f(x_1) < 0$ (or) $f(x_1) > 0$

Let us consider that $f(x_1) < 0$ & $f(a) > 0$ Since $f(x_1)\, f(a) < 0$ $\Rightarrow$ Root lies between x_0 and a $\Rightarrow x_2 = \frac{x_1+a}{2}$	Let us consider that $f(x_1) > 0$ & $f(x_0) < 0$ Since $f(x_1)\, f(x_0) < 0$ $\Rightarrow$ Root lies between x_1 and x_0 $\Rightarrow x_2 = \frac{x_1+x_0}{2}$	Let us consider that $f(x_1) > 0$ & $f(b) < 0$ Since $f(x_1)\, f(b) < 0$ $\Rightarrow$ Root lies between x_0 and b $\Rightarrow x_2 = \frac{x_1+b}{2}$	Let us consider that $f(x_1) < 0$ & $f(x_0) > 0$ Since $f(x_1)\, f(x_0) < 0$ $\Rightarrow$ Root lies between x_1 and x_0 $\Rightarrow x_2 = \frac{x_1+x_0}{2}$

Here, the logic is that, we have to select first negative or first positive from bottom approximations only, but not from top. i.e. the approximation which we have recently found should be selected.

If $f(x_1) = 0$, then x_1 is root of the given equation. Otherwise repeat above process until we obtain solution of the given equation.

Regular Folsi Method (Or) Method of False position

Let us consider that $y = f(x)$ be the given equation.

Let us choose two points a and b in such a way that $f(a)$ and $f(b)$ are of opposite signs.

i.e. $f(x_0).f(x_1) < 0$

Consider $f(a) > 0, f(b) < 0$, so that the graph $y = f(x)$ crosses X-axis in between a and b . then, there exists a root of the given equation in between a and b.

Let us define a straight line joining the points $A(a, f(a)),\ B(b, f(b))$, then the equation of straight line is given by $\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$ -----> (I)

This method consists of replacing the part of the curve by means of a straight line and then takes the point of intersection of the straight line with X-axis, which give an approximation to the required root.

In the present case, it can be obtained by substituting $y = 0$ in equation I, and it is denoted by x_0 .

From (I) $\Longrightarrow\ y - f(a) = \frac{f(b)-f(a)}{b-a}\ (x - a)$

Now, any point on X-axis $\Longrightarrow y = 0$ and let initial approximation be x_0 i.e. $x = x_0$

$\therefore$ (I) $\Longrightarrow 0 - f(a) = \frac{f(b)-f(a)}{b-a}\ (x_0 - a)$

$$\Longrightarrow\ x_0 - a = \frac{-f(a)\ (b-a)}{f(b)-f(a)}$$

$$\Longrightarrow\ x_0 = a - \frac{f(a)\ (b-a)}{f(b)-f(a)}$$

$$\Longrightarrow\ x_0 = \frac{a\ f(b)-b\ f(a)}{f(b)-f(a)}$$ ----> (II)

If $f(x_0) = 0$, then x_0 is root of the given equation.

If $f(x_0) \neq 0$, then either $f(x_0) < 0$ (or) $f(x_0) > 0$

Case (i): Let us consider that $f(x_0) < 0$

We know that $f(a) > 0$ and $f(x_0) < 0$

Hence root of the given equation lies between a and x_0 .

In order to obtain next approximation x_1, replace b with x_0 in equation (II)

Hence $x_1 = \frac{a\, f(x_0) - x_0\, f(a)}{f(x_0) - f(a)}$

Case (ii): Let us consider that $f(x_0) > 0$

We know that $f(b) < 0$ and $f(x_0) > 0$

Hence root of the given equation lies between b and x_0 .

In order to obtain next approximation x_1, replace a with x_0 in equation (II)

Hence $x_1 = \frac{x_0\, f(b) - b\, f(x_0)}{f(b) - f(x_0)}$

If $f(x_1) = 0$, then x_1 is root of the given equation. Otherwise repeat above process until we obtain a root of the given equation to the desired accuracy.

Newton Raphson Method

Let us consider that $f(x) = 0$ be the given equation.

Let us choose initial approximation to be x_0 .

Let us assume that $x_1 = x_0 + h$ be exact root of $f(x) = 0$ where $h \ll 0$, so that $f(x_0 + h) = 0$

Expanding above relation by means of Taylor's expansion method, we get

$$f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ ... = 0$$

Since h is very small, h and higher powers of h are neglected. Then the above relation becomes

$$f(x_0) + h\, f'(x_0) = 0 \implies h = -\frac{f(x_0)}{f'(x_0)}$$

Hence $x_1 = x_0 + h$

$$\implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $f(x_1) = 0$, then x_1 is root of the given equation. Otherwise repeat above process until we obtain a root of the given equation to the desired accuracy.

Successive approximations are given by

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ and so on. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Iteration Method

Let us consider that $f(x) = 0$ be the given equation.

Let us choose initial approximation to be x_0 .

Rewrite $f(x) = 0$ as $x = \varphi(x)$ such that $|\varphi'(x_0)| < 0$.

Then successive approximations are given by $x_1 = \varphi(x_0)$

$$x_2 = \varphi(x_1)$$

$$\vdots$$

$$x_n = \varphi(x_{n-1})$$

Repeat the above process until we get successive approximation equal, which will gives the required root of the given equation.

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