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Elements of Statistical Mechanics

Question: Discuss about principles of Maxwell-Boltzmann Statistics?

Answer: Maxwell – Boltzmann Statistics

In classical mechanics all the particles (fundamental and composite particles, atoms, molecules, electrons, etc.). In the system are considered distinguishable. This means that one can label and track each individual particle in a system. As a consequence changing the position of any two particles in the system leads to a completely different configuration of the entire system. Further more there is no restriction on placing more than one particle in any given state accessible to the system. Classical statistics is called Maxwell – Boltzmann Statistics (or M – B Statistics).

The fundamental feature of quantum mechanics that distinguishes it from classical mechanics is that particles of a particular type are indistinguishable from one another. This means that in an assembly consisting of similar particles, interchanging any two particles does not need to a new configuration of the system (in the language of quantum mechanics; the wave function of the system is invariant with respect to the interchange of the constituent particles.). In case of a system consisting of particles belonging to different nature (for example electrons and protons), the wave function of the system is invariant separately for the assembly of the two particles.

While this difference between classical and quantum description of systems is fundamental to all of quantum statistics, it is further divided into the following two classes on the basis of symmetry of the system.

Maxwell – Boltzmann Statistics expression:

Maxwell – Boltzmann statistics describes the statistical distribution of material particles over various energy states in thermal equilibrium, when the temperature is high enough and density is low enough to render quantum effects negligible.

The expected number of particles with energy ε_i for Maxwell – Boltzmann statistics is N_i

Where

$$\frac{N_i}{N} = \frac{g_i}{\exp\left(\frac{E_i - E_F}{kT}\right)} = \frac{g_i \exp\left(\frac{-E_i}{kT}\right)}{Z}$$

Where: N_i is the number of particles in state i

E_i is the energy of the i -th state

g_i is the degeneracy of energy level i , the number of particle's states (excluding the free particle state) with energy ε_i

E_F is the chemical potential

K is the Boltzmann's constant

T is absolute temperature

N is the total number of particles, $N = \sum_i N_i$

Z is the partition function, $Z = \sum_i g_i \exp\left(\frac{-E_i}{kT}\right)$

Equivalently, the distribution is sometimes expressed as

$$\frac{N_i}{N} = \frac{1}{\exp\left(\frac{E_i - E_F}{kT}\right)} = \frac{g_i \exp\left(\frac{-E_i}{kT}\right)}{Z}$$

Where the index i now specifies a particular state than the set of all states with energy ϵ_i .

Question: Explain the principle of Bose-Einstein Statistics?

Answer: Bose – Einstein Statistics

In Bose –Einstein statistics (B- E Statistics) interchanging any two particles of the system leaves the resultant system in a symmetric state. That is, the wave function of the system before interchanging equals the wave function of the system after interchanging.

It is important to emphasize that the wave function of the system has not changed itself. This has very important consequences on the state of the system: There is no restriction to the number of particles that can be placed in a single state (accessible to the system). It is found that the particles that obey Bose – Einstein statistics are the ones which have integer spins, which are therefore called bosons (named for Bose). Example of the bosons include photons and helium – 4 atoms. One type of system obeying B-E statistics in the Bose – Einstein condensate where all particles of the assembly exist in the same state.

Question: Discuss about Fermi-Dirac statistics and Fermi-Dirac distribution ?

Answer: Fermi – Dirac statistics

In Fermi – Dirac statistics (F- D statistics) interchanging any two particles of the system leaves the resultant system in an antisymmetric state. That is, the wavefunction of the system before interchanging is the wavefunction of the system after interchanging, with an overall minus sign

Again it should be noted that the wavefunction of the system itself does not change. The consequence of the negative sign on the Fermi – Dirac statistics can be understood in the following way:

Suppose that the particles that are interchanged belong to the same state. Since the particles are considered indistinguishable from one another then changing the coordinates of the particles should not have any change on the system's wavefunction (because by

our assumption the particles are in the same state). Therefore, the wavefunction before interchanging similar states equals the wavefunction after interchanging similar states.

Combining (or adding, literally speaking) the above statement with the fundamental symmetry of the Fermi –Dirac system leads us to conclude that the wavefunction of the system before interchanging equals zero.

This shows that in Fermi –Dirac statistics, more than one particle cannot occupy a single state accessible to the system. This is called Pauli's exclusion principle.

It is found that particles with half – integral spin (or fermions) obey the Fermi – Dirac statistics. This includes, electrons, protons, Helium – 3 etc.

Fermi – Dirac and Bose – Einstein statistics apply when quantum effects are important and the particles are indistinguishable. Quantum effects appear if the concentration of particles $(N/V) \geq n_q$. here n_q is the quantum concentration, for which the interparticle distance is equal to the thermal de Broglie wavelength, so that the wavefunctions of the particles are touching but not overlapping. Fermi- Dirac statistics apply to fermions (particles that obey the Pauli exclusion principle), and Bose – Einstein statistics apply to bosons. As the quantum concentration depends on temperature; most systems at high temperatures obey the classical (Maxwell – Boltzmann) limit unless they have a very high density, as for a white dwarf. Both Fermi – Dirac and Bose- Einstein become Maxwell – Boltzmann statistics at high temperature or at low concentration.

Maxwell – Boltzmann statistics are often described as the statistics of distinguishable classical particles. In other words the configuration of particle A in state 1 and particle B in state 2 is different from the case where particle B is in state 1 and particle A is in state 2. This assumption leads to the proper (Boltzmann) distribution of particles in the energy states, but yields non – physical results for the entropy, as embodied in the Gibbs paradox. This problem disappears when it is realized that all particles are in fact indistinguishable. Both of these distributions approach the Maxwell – Boltzmann distribution in the limit of high temperature and low density, without the need for any ad hoc assumptions. Maxwell – Boltzmann statistics are particularly useful for studying gases. Fermi – Dirac statistics are most often used for the study of electrons in solids. As such, they form the basis of semiconductor device theory and electronics.

Fermi – Dirac distribution:

For a system of identical fermions, the average number of fermions in a single – particle state i , is given by the Fermi –Dirac (F-D) distribution.

$$n_i = \frac{1}{\exp\left(\frac{E_i - E_F}{kT}\right) + 1}$$

Where K is Boltzmann's constant, T is the absolute temperature, E_i is the energy of the single – particle state I , and E_F is the chemical potential. For the case of electrons in a semiconductor, E_F is also called the Fermi level.

The F-D distribution is valid only if the number of fermions in the system is large enough so that adding one more fermion to the system has negligible effect on E_F . Since the F-D distribution was derived using the Pauli exclusion principle, a result is that $0 < n_i < 1$.

Question: Discuss about the properties of Fermi-Dirac Statistics ?

Answer: Properties of the Fermi – Dirac statistics

In a semiconductor, the probability of occupancy of states by electrons is given by the Fermi – Dirac distribution function.

$$F(E) = P_e(F) = P_{FD} = \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)}$$

(8.1)

- i. The distribution function $P_e(E)$ is valid only in equilibrium.
- ii. E_F denotes an energy level and is called Fermi energy level. It is strictly valid in equilibrium.
- iii. It is applicable for all insulators, semiconductors and metals.

- iv. It considers all electrons in semiconductors and not merely electrons in a band.
- v. An empty electron state is called a hole. The probability of occupancy of states by holes is given by $P_h(E) = 1 - P_e(E)$

From the equation (8.1)

$$P_h(E) = 1 - \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)}$$

$$P_h(E) = \frac{1 + \exp\left(\frac{E - E_F}{kT}\right) - 1}{1 + \exp\left(\frac{E - E_F}{K_B T}\right)}$$

$$P_h(E) = 1 - \frac{1}{1 + \exp\left(\frac{-E + E_F}{kT}\right) + 1}$$

$$P_h(E) = \frac{1}{1 + \exp\left(\frac{E_F - E}{kT}\right)}$$

vi. At $E = E_F$ $P_h(E) = P_e(E) = \frac{1}{1+1} = \frac{1}{2}$

vii. At 0 K, if $E > E_F$, $P_e(E) = \frac{1}{1+e^x} = \frac{1}{1+\infty} = 0$

If $E \geq E_F$, $P_e(E) = \frac{1}{1+e^x} = \frac{1}{1+0} = 1$

This means that at absolute zero temperature, energy states up to the Fermi level are completely occupied by electrons, and the levels above E_F are empty i.e.

$$\text{For all, } E - E_F \gg kT, \exp\left(\frac{E - E_T}{kT}\right) \ll 1, \text{ hence } \begin{cases} P_e(E) = 0 \\ P_h(E) = 1 \end{cases}$$

$$\text{For all, } E - E_F \gg kT, \exp\left(\frac{E - E_T}{kT}\right) \ll 1, \text{ hence } \begin{cases} P_e(E) = 1 \\ P_h(E) = 0 \end{cases}$$

- viii. the functions $P_e(E)$ and $P_h(E)$ possess a certain symmetry about the Fermi level as shown in Fig. 8.1. consider a level $(E_p + \delta E)$ just above the Fermi level. The electron occupancy at this level is

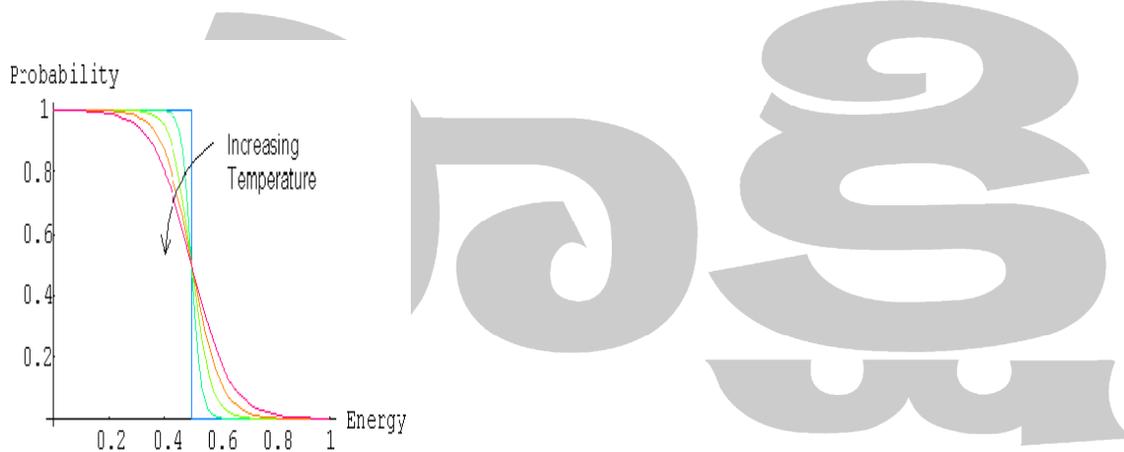


fig 1: Fermi –Dirac distribution of electrons

$$P_e(E_F + \delta E) = \frac{1}{1 + \exp\left(\frac{E_F - E_F + \delta E}{kT}\right)}$$

$$P_e(E_F + \delta E) = \frac{1}{1 + \exp\left(\frac{\delta E}{kT}\right)}$$

Similarly the hole occupancy at a level $E_F - \delta E$ just below the Fermi level is

$$P_h(E_F - \delta E) = \frac{1}{1 + \exp\left(\frac{E_F - E_F + \delta E}{kT}\right)}$$

$$P_h(E_F - \delta E) = \frac{1}{1 + \exp\left(\frac{\delta E}{kT}\right)}$$

Hence,

$$P_e(E_F + \delta E) = P_h(E_F - \delta E)$$

Thus the hole occupancy probability, at a level $E_F - \delta E$ below the Fermi level is equal to the electron occupancy probability, at a level $E + \delta E$ above the Fermi level.

- ix. The number of electrons per unit volume at an energy level E is given by
 $n = \{ \text{Density of states at } E \} \{ \text{The probability of occupation of electrons in } E \}$
 $n(E) = g_e(E) P_e(E)$ (8.3)

similar for the number of holes per unit volume at an energy level E is given by

$$P(E) = g_h(E) P_h(E)$$
 (8.4)

- x. The carrier densities n_0 and p_0 in the range E_1 and E_2 in equilibrium can be calculated as

$$n_0 = \int_{E_1}^{E_2} n(E) dE = \int_{E_1}^{E_2} g_e(E) P_e(E) dE$$
 (8.5)

And

$$p_0 = \int_{E_1}^{E_2} P(E) dE = \int_{E_1}^{E_2} g_h(E) P_h(E) dE$$
 (8.6)

- xi. The level E_F in a connected system at equilibrium remains constant. Consider connected system of S_1 and S_2 with $E_{F1} = E_{F2}$.

Then the number of particles in S_1 is

$$\begin{aligned}
 n_1(E) &= g_1(E) p_{e1}(E) \\
 &= g_1(E) P_{FD1}(E)
 \end{aligned}$$

And the number of available states in S_2 is

$$\begin{aligned}
 n_2(E) &= g_2(E) [1 - p_{e2}(E)] \\
 &= g_2(E) [1 - P_{FD2}(E)]
 \end{aligned}$$

Hence, the number of electrons flowing from S_1 to S_2 is

$$n_{12} = C_0 g_1(E) P_{FD1}(E) g_2(E) [1 - P_{FD2}(E)]$$

And the number of Fermi ions tending to move from S_2 to S_1 is

$$n_{21} = C_0 g_2(E) P_{FD2}(E) g_1(E) [1 - P_{FD1}(E)]$$

Where C_0 is proportional constant.

In equilibrium $n_{12} = n_{21}$, therefore

$$\frac{P_{FD1}(E)}{1 - P_{FD1}(E)} = \frac{P_{FD2}(E)}{1 - P_{FD2}(E)}$$

Hence, $E_{F1} = E_{F2}$

Thus, in equilibrium, Fermi level is a horizontal line in the combined system of S_1 and S_2 .

Question: Discuss about the properties of Photon gas ?

Answer: Photon Gas:

A photon gas is a gas – like collection of photons, which has many of the same properties of a conventional gas like hydrogen or neon – including pressure, temperature and entropy. The most common example of a photon gas in equilibrium is black body radiation.

A massive ideal gas with only one type of particle is uniquely described by three state functions such as the temperature, volume and the number of particles. However for a black body, the energy distribution is established by the interaction of the photons with matter, usually the walls of the container. In this interaction the number of photons is not conserved. As a result the chemical potential of the black body photon gas is zero. The number of state functions needed to describe a black body state thus reduced from three to two (e.g. temperature and volume).

In a gas with massive particles, the energy of the particles is distributed according to a Maxwell – Boltzmann distribution. This distribution is established as the particles collide with each other, enhancing energy (and momentum) in the process. In a photon gas, there will also be an equilibrium distribution, but photons do not collide with each other (except under very extreme conditions) so that the equilibrium distribution must be established by other means. The most common way that an equilibrium distribution is established is by the interaction of the photons with matter. If the photons are absorbed and emitted by the walls of the system containing the photon gas, and the walls are at a particular temperature, when the equilibrium distribution for the photons will be black body distribution at that temperature.

A very important difference between a gas of massive particles and a photon gas with black body distribution is that the number of photons in the system is not conserved. A photon may collide with an electron in the wall, exciting it to a higher energy state, removing a photon from the photon gas. The electron may drop back to its lower level a series of steps, each one of which releases an individual photon back into the photon gas. Although the sum of the energies of the emitted photons are the same as the absorbed photon, the number of emitted photons will vary. It can be shown that, as a result of this lack of constraint on the number of photons in the system, the chemical potential of the photons must be zero for blackbody radiation.

The thermodynamics of a black body photon gas may be derived using quantum mechanical arguments. The derivation yields the spectral energy distribution u which is the energy per unit volume per unit frequency interval:

$$u(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

Where h is Planck's constant, c is the speed of light, ν is the frequency, k is the Boltzmann's constant, and T is temperature.

Question: What is Wien's displacement law?

Answer: Wien's Displacement Law:

Wien's displacement law states that the black body curve at any temperature is uniquely determined from the black body curve at any other temperature by displacing, or shifting, the wavelength. The average thermal energy/ frequency in each mode with frequency ν is only a function of ν/T . Restated in terms of the wavelength $\lambda = c/\nu$, the distribution at corresponding wavelengths are related, where corresponding wavelengths are at locations proportional to $1/T$.

From this law, it follows that there is an inverse relationship between the wavelength of the peak of the emission of a black body and its temperature, and this less powerful consequence is often also called Wien's displacement law is

$$\lambda_{\max} = \frac{b}{T}$$

Where

λ_{\max} is the peak wavelength in meters

T is the temperature of the black body in kelvins (K), and

b is a constant of proportionality, called Wien's displacement constant.

Question: What is Rayleigh – Jeans law?

Answer: Rayleigh – Jeans Law:

The Rayleigh – Jeans law, first proposed in the early 20th century, attempts to describe the spectral radiance of electromagnetic radiation at all wavelengths from a black body at a given temperature through classical arguments. For wavelength λ , it is ;

$$B_{\lambda}(T) = \frac{2ckT}{\lambda^4}$$

Where c is the speed of light, k is Boltzmann's constant and T is temperature in kelvins. For frequency ν , it is;

$$B_{\nu}(T) = \frac{2\nu^2kT}{c^2}$$

The Rayleigh – Jeans expression agrees with experimental results at large wavelengths (or, equivalently, low frequencies) but strongly disagrees at short wavelengths (or high frequencies). This inconsistency is commonly known as the ultraviolet catastrophe.

Question: Explain about the Planck's law of Black Body radiation ?

Answer: Planck's Law of Black Body Radiation:

Planck's law describes the spectral radiance of electromagnetic radiation at all wavelengths from a black body at temperature T . as a function of frequency ν , planck's law is written as:

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

The function represents emitted power per unit area of emitting surface, per unit solid angle, and per unit frequency. Sometimes, planck's law is written as an

expression $u(\nu, T) = \pi I(\nu, T)$ for emitted power integrated over all solid angles. In other cases, it is written as $u(\nu, T) = 4\pi I(\nu, T)/c$ for energy per unit volume.

The function $I(\nu, T)$ peaks for $h\nu = 2.82 kT$. It falls off exponentially at higher frequencies and polynomially at lower. As a function of wavelength λ , plank's law written (for unit solid angle) as:

$$I'(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}$$

This function peaks for $hc = 4.97 \lambda kT$, a factor of 1.76 shorter in wavelength (higher in frequency) than the frequency peak. It is the more commonly used peak in Wien's displacement law.

The radiance emitted over a frequency range $[\nu_1, \nu_2]$ or a wavelength range $[\lambda_2, \lambda_1] = [c/\nu_2, c/\nu_1]$ can be obtained by integrating the representing the respective functions.

$$\int_{\nu_1}^{\nu_2} I(\nu, T) d\nu = \int_{\lambda_2}^{\lambda_1} I'(\lambda, T) d\lambda$$

The order of the integration limits is reversed because increasing frequencies correspond to decreasing wavelengths.

The wavelength is related to the frequency by:

$$\lambda = \frac{c}{\nu}$$

This is sometimes written in terms of spectral energy density.

$$u(\nu, T) = \frac{4\pi}{c} I(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

Which has units of energy per unit volume per unit frequency (joule per cubic meter per hertz). Integrated over frequency, this expression yields the total energy density, the radiation field of a black body may be thought of as a photon gas, in which case this energy density would be one of the thermodynamic parameters of that gas.

The spectral energy density can also be expressed as a function of wavelength:

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e\left(\frac{hc}{\lambda kT}\right) - 1}$$

Max Planck originally produced this law in 1900 (published in 1918) in an attempt to improve upon the Wien approximation, published in 1896 by Wilhelm Wien, which fit the experimental data at short wavelengths (high frequencies) but deviated from it at long wavelengths (low frequencies). The Rayleigh – Jeans law (first published in 1900 by Rayleigh in 1900) fit well in the complementary domain (long wavelength, low frequency). Planck found that the above function, Planck's function, fitted the data for all wavelengths remarkably well. In constructing a derivation of this law, he considered the possible ways of distributing electromagnetic energy over the different modes of charged oscillators in matter. Planck's law emerged when he assumed that the energy of these oscillators was limited to a set of discrete, integer multiples of a fundamental unit of energy, E , proportional to the oscillation frequency ν :

$$E = h\nu$$

Planck made this quantization assumption five years before Albert Einstein hypothesized the existence of photons as a means of explaining the photoelectric effect. At the time, Planck believed that the quantization applied only to the tiny oscillators that were thought to exist in the walls of the cavity (what we now know to be atoms), and made no assumption that light itself propagates in discrete bundles or packets of energy. Moreover, Planck did not attribute any physical significance to this assumption, but rather believed that it was merely a mathematical device that enabled him to derive a single expression for the black body spectrum that matched the empirical data at all wavelengths.

Although Planck's formula predicts that a black body will radiate energy at all frequencies, the formula is only practically applicable when many photons are being measured. For example, a black body at room temperature (300 kelvin) with one square meter or surface area will emit a photon in the visible range once about every thousand years or so, meaning that for most practical purpose, a black body at room temperature does not emit in the visible range. Significance of this fact for the derivation of Planck's law from experiment data, and for the substantiation of the law by the data is discussed.

Ultimately, Planck's assumption of energy quantization and Einstein's photon hypothesis became the fundamental basis for the development of quantum mechanics.

Question: What is concept of free electron gas?

Answer: Concept of Electron Gas and explain about root mean square (rms) velocity:

In 1900, Drude and Lorentz proposed the classical free electron theory. According to this theory, metal is an aggregate of fixed positive charges and negative free electron gas and the free electron gas is obeying the laws of Kinetic theory of gases. This means electrons can be assigned a mean free path, a mean collision time and average speed. The salient features of classical free electron theory are given below.

- i. The free electrons, in all metals behave as the molecules of a gas in a container.
- ii. The mutual repulsion among the electrons is ignored, so that they move in all the directions with all the possible velocities.
- iii. As the electrons move randomly with rms velocity, the resultant velocity of them in a particular direction will be zero.
- iv. If the potential difference is created across the metal then free electrons slowly drift towards the positive potential or in the opposite direction to the applied electric field even though they collide continually with one another and fixed positive metal ions in the lattice. The acquired velocity by the electrons is called drift velocity and is represented by v_d .

Root Mean Square (rms) velocity (\bar{C})

According to equipartition theory, the expression for the pressure on the electron can be written as $P = \frac{1}{3}\rho\bar{C}^2$

$$P = \frac{1}{3}mn\bar{C}^2 \quad (8.7)$$

Where P is pressure on the electrons, m is mass of the electrons, n is the number electrons per unit volume, \bar{C} is the root mean square velocity of the electrons and ρ is density of electrons.

If a molar volume of substance is taken then the number of molecules present will be equal to avagadro number N_A and its volume is V_m , then

$$P = \frac{1}{3}m\frac{N_A}{V_m}\bar{C}^2$$

$$P V_m = \frac{1}{3}mN_A\bar{C}^2 = RT \quad (\text{since } PV_m = RT)$$

$$m\bar{C}^2 = \frac{3RT}{N_A}$$

$$\frac{1}{2}m\bar{C}^2 = \frac{3}{2}K_B T = KE \quad (\text{since, } N_A K_B = R) \quad (8.8)$$

$$\bar{C}^2 = \frac{3K_B T}{m}$$

$$\bar{C} = \sqrt{\frac{3K_B T}{m}} \quad (8.9)$$

$$\bar{C} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{9.1 \times 10^{-31}}}$$

$$\bar{C} = 1.16 \times 10^5 \text{ m/s}$$

Hence, the free electron gas is moving with $\bar{C} = 1.16 \times 10^5 \text{ m/s}$ velocity. Thus it is in motion, the resultant displacement will be zero as it is in random motion and no currents are present in the metal when there are no external forces.

Question: Explain Fermi level and Fermi energy with the help of suitable diagram?

Answer: Fermi Energy:

The Fermi energy is a concept in quantum mechanics usually referring to the energy of the highest occupied quantum state in a system of fermions at absolute zero temperature. This article requires a basic knowledge of quantum mechanics.

Note that the term Fermi energy is often, confusingly, used to describe a different but closely related concept, the Fermi level (also called chemical potential). The Fermi energy and chemical potential are the same at absolute zero, but differ at other temperatures, as described below.

In quantum mechanics, a group of particles known as fermions (for example, electrons, photons and neutrons are fermions) obey the Pauli exclusion principle, which states that no two fermions can occupy the same quantum state. The states are labeled by a set of quantum numbers. In a system containing many fermions (like electrons in a metal) each fermion will have a different set of quantum numbers. To determine the lowest energy

system of fermions can have, we first group the states into sets with equal energy, and order these sets by increasing energy. Starting with an empty system, we then add particles one at a time, consecutively filling up the unoccupied quantum states with the lowest energy. When all the particles have been put in, the Fermi energy is the energy of the highest occupied state. What this means is that even if we have extracted all possible energy from a metal by cooling it down to near absolute zero temperature (0 kelvins), the electrons in the metal are still moving around; the fastest ones would be moving at a velocity that corresponds to a kinetic energy equal to the Fermi energy. This is the Fermi velocity. The Fermi energy is one of the important concepts of condensed matter physics. It is used, for example, to describe metals, insulators and semiconductors. It is a very important quantity in the physics of superconductors, in the physics of quantum liquids like low temperature helium (both normal and superfluid ^3He), and it is quite important to nuclear physics and to understand the stability of white dwarf stars against gravitational collapse.

The Fermi energy (E_F) of a system of non – interacting fermions is the increase in the ground state energy when exactly one particle is added to the system. It can also be interpreted as the maximum energy of an individual fermion in this ground state. The chemical potential at zero temperature is equal to the Fermi energy.

The probability to find an electron in an energy state of energy E can be expressed as

$$F(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)} \quad (8.10)$$

Where $F(E)$ is called the Fermi Dirac distribution function. E is the energy level occupied by the electron and E_F is the Fermi level and is constant for a particular system.

The Fermi level is a boundary energy level which separates the filled energy states and empty energy states at 0 K. The energy of the highest filled state at 0 K is called the Fermi energy level is known as Fermi level. It is shown in Fig. 8.2.

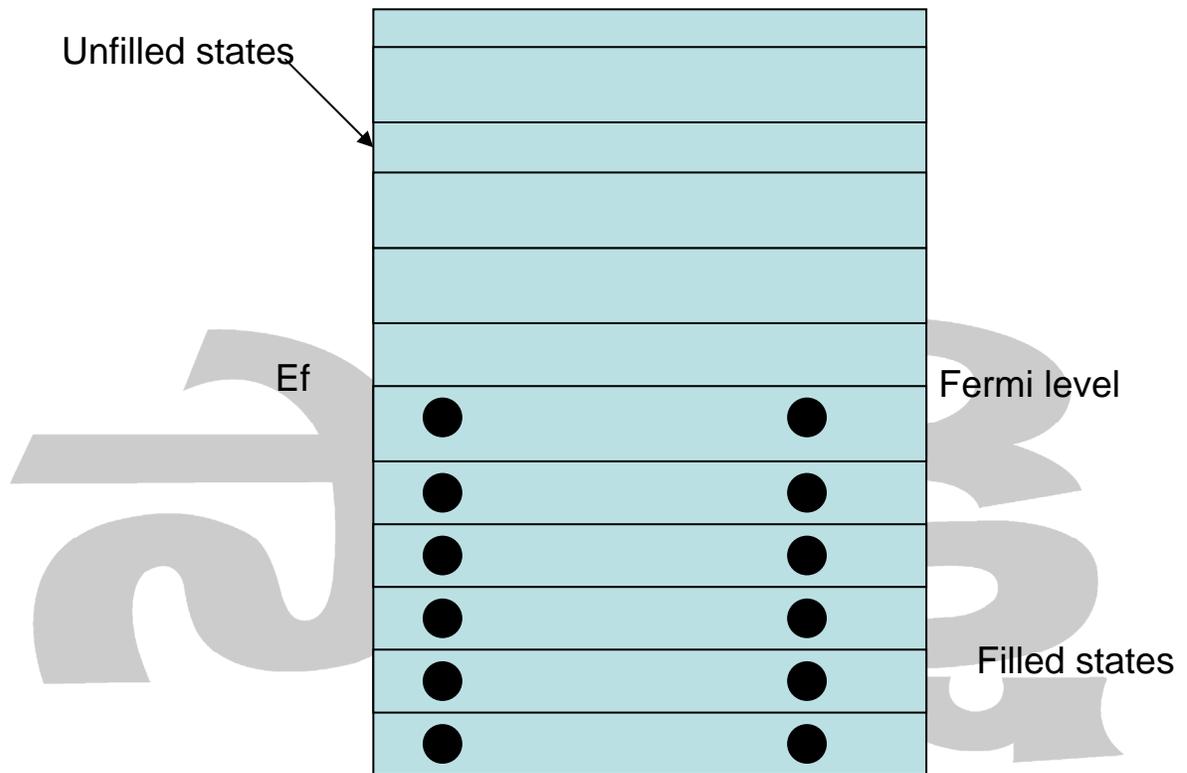


Fig: 8.2: Highest filled energy level at 0 K.

Fermi – Dirac distribution curve at 0 K and at different temperatures is shown in Fig. 8.3.

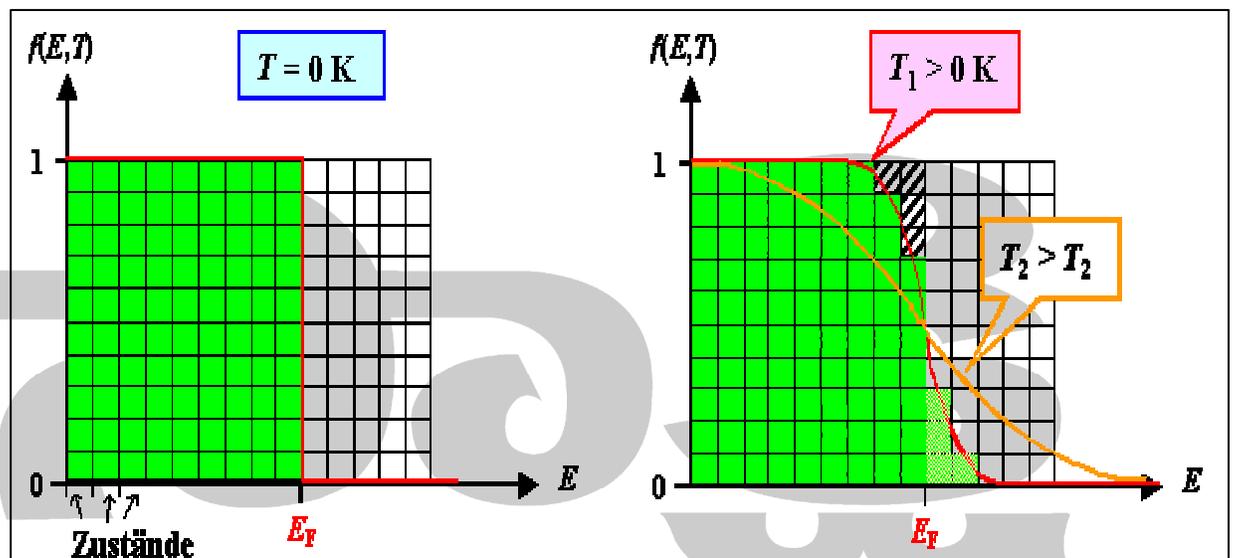


Fig: 8.3: Fermi – Dirac distribution curve at 0 K and at different temperatures. At 0 K, the Fermi – Dirac distribution of electrons can be understood mathematically from the following two cases,

Case (i) if $E > E_F$ then $F(E) = 0$

It indicates that the energy levels above the Fermi level are empty.

Cases (ii) if $E < E_F$ then $F(E) = 1$

It indicates that the energy levels below the fermi level are full with electrons.

The variation of Fermi – Dirac distribution function with temperature is shown in Fig. 8.3. From this figure, it can be observed that the probability to find an electron

decreases below the Fermi level and increases above the Fermi level as temperature increase. And there exists a two – fold symmetry un the probability curves about the Fermi level.

The product of Fermi – Dirac distribution function and density of states gives the number of electrons per unit volume

$$\text{i.e.,} \quad n = \int_0^{E_F} F(E) g(E) dE \quad (8.11)$$

substituting F(E) and G(E) values in equation 8.11

$$n = \int_0^{E_F} \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)} \frac{\pi \left(\frac{8m}{h^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}} dE}{}$$

If a metal is taken at 0 K then energy levels below Fermi level are filled by electrons and above Fermi level are empty. If T = 0 K and E < E_F, then

$$n = \int_0^{E_F} \frac{\pi \left(\frac{8m}{h^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}} dE}{}$$

$$n = \frac{\pi \left(\frac{8m}{h^2}\right)^{\frac{3}{2}}}{2} \int_0^{E_F} E^{\frac{1}{2}} dE$$

$$n = \frac{\pi}{2} \left(\frac{8m}{h^2} \right)^{\frac{3}{2}} \left| \frac{E^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^{E_F}$$

$$E_F = \left(\frac{3}{\pi} \right)^{\frac{2}{3}} \frac{h^2}{8m} n^{\frac{2}{3}}$$

$$E_F = \left(\frac{3}{\pi} \right)^{\frac{2}{3}} \frac{h^2}{8m^2} n^{\frac{2}{3}}$$

Substituting the values of h, n, m and π , the Fermi energy,

$$E_F = 3.62 \times 10^{-19} n^{2/3} \text{ eV}$$

Question: Give a short notes on Density of states in an atom?

Answer: Density of States:

The number of electrons present per unit volume in an energy level at a given temperature is equal to the product of density of states (number of energy levels per unit volume) and Fermi – Dirac distribution function (the probability to find an electron). Therefore, to calculate the number of electrons in an energy level at a given temperature, it is must to know the number of energy states per unit volume.

The number of energy states with a particular energy value E is depending on how many combinations of the quantum numbers resulting in the same value n. the energy levels appear continuum inside the space of an atom, therefore, let n_x , n_y , and n_z ($n^2 = n_x^2 + n_y^2 + n_z^2$) in the three dimensional space. In this space every integer specifies an energy level i.e., each unit cube contains exactly one state. Hence the number of energy states in any volume in units of cubes of lattice parameters.

Consider a space of radius n and another sphere of radius (n + dn) in which energy values are E and E + dE respectively as shown in Fig. 8.4. hence, the number of energy states available in the sphere of radius n is

$$\frac{1}{8} \left(\frac{4}{3} \pi n^3 \right) \quad (8.12)$$

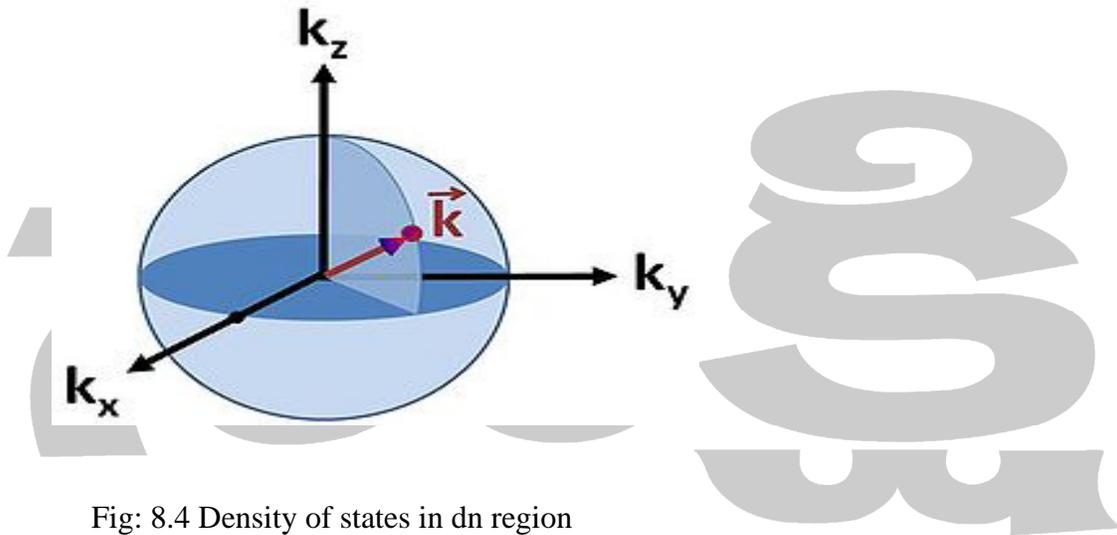


Fig: 8.4 Density of states in dn region

And the number of energy states in the sphere of radius $(n + dn)$ is

$$\frac{1}{8} \left(\frac{4}{3} \pi (n + dn)^3 \right) \quad (8.13)$$

E and dE are the energy values of above two regions respectively. From the equations (8.12) and (8.13), the number of energy states available in dn (or) dE region,

$$g'(E)dE = \frac{1}{8} \left[\frac{4}{3} \pi (n + dn)^3 \right] - \frac{1}{8} \left[\frac{4}{3} \pi n^3 \right]$$

$$= \frac{\pi}{6} [(n + dn)^3 - n^3]$$

$$g'(E)dE = \frac{\pi}{6} [n^3 + dn^3 + 3n^2 dn + 3ndn^2 - n^3]$$

Neglecting higher order terms,

$$g'(E)dE = \frac{\pi}{6} [3n^2 dn]$$

$$g'(E)dE = \frac{\pi}{2} [n(ndn)] \quad (8.14)$$

The expression for n th energy level can be written as

$$E = \frac{n^2 h^2}{8ma^2} \quad (8.15)$$

$$n = \left[\frac{8ma^2}{h^2} E \right]^{\frac{1}{2}} \quad (8.16)$$

Differentiating equation (8.16), taking n and E are variables,

$$2ndn = \frac{8ma^2}{h^2} dE$$

$$ndn = \frac{1}{2} \left(\frac{8ma^2}{h^2} \right) dE \quad (8.17)$$

Substituting equations (8.16) and (8.17) in equation (8.14)

$$g'(E)dE = \frac{\pi}{2} \left(\frac{8ma^2}{h^2} \right)^{\frac{1}{2}} E^{\frac{1}{2}} \cdot \frac{1}{2} \left[\frac{8ma^2}{h^2} \right] dE$$

$$g'(E)dE = \frac{\pi}{4} \left(\frac{8ma^2}{h^2} \right)^{\frac{1}{2}} E^{\frac{1}{2}} dE$$

According to Pauli's exclusion principle each energy level contains two electrons. This means each energy level will have two sub energy levels, therefore, the above equation should be multiplied by 2.

$$g'(E)dE = 2 \times \frac{\pi}{4} \left(\frac{8ma^2}{h^2} \right)^{\frac{3}{2}} E^{\frac{1}{2}} dE$$

$$g'(E)dE = \frac{\pi}{2} \left(\frac{8ma^2}{h^2} \right)^{\frac{3}{2}} E^{\frac{1}{2}} dE$$

$$g'(E)dE = \frac{\pi}{2} \left(\frac{8ma^2}{h^2} \right)^{\frac{3}{2}} a^3 E^{\frac{1}{2}} dE \quad (8.18)$$

Density of states,

$$g(E)dE = \frac{g'(E)}{V} dE$$

If length of the energy level is then its volume $V = a^3$. hence, density of states,

$$g(E)dE = \frac{\pi}{2} \left(\frac{8m}{h^2} \right)^{\frac{3}{2}} E^{\frac{1}{2}} dE \quad (8.19)$$

Problems:

1. Find the temperature at which there is 1% probability that a state with an energy 0.5 eV above Fermi energy will be occupied.

(Ans: $T = 1270\text{K}$)

2. At what temperature we can expect a 10% probability that electron in silver have an energy which is 1% above the Fermi energy? The Fermi energy of silver is 5.5 eV.
(Ans: $T = 290.2\text{K}$)

3. compute the average kinetic energy of a gas molecule at 30°C . Express the result in electron volts. If the gas is hydrogen. What is the order of magnitude of the velocity of molecules at 30°C ?

(Ans: $\bar{c} = 6104.69\text{ m/s}$)

4. calculate the velocity of an electron with kinetic energy of 10 electron volt; what is the velocity of a proton with kinetic energy of 10 electron volt?

$$(\text{Ans: } c_p = \sqrt{\frac{2 \times 10 \times 1.6 \times 10^{19} \text{ J}}{9.1 \times 10^{-31} \text{ kg}}}; c_p = 19.1616 \times 10^{12} \text{ m/s})$$

5. Find the drift velocity of free electron in a copper wire of cross sectional area 10 mm^2 when the wire carries a current of 100A. assume that each copper atom contributes one electron to the electron gas.

(Ans: $u_d = 7.35294 \times 10^{-4} \text{ m/s}$)

6. Calculate the energy of a photon of sodium light of wavelength $5893 \times 10^{-10}\text{ m}$ in (a) joules and (b) electron – volts (eV). Take $h = 6.62 \times 10^{-34} \text{ J-s}$ and $c = 3 \times 10^8 \text{ m/s}$.

(Ans: 2.11eV)

7. A radio transmitter operates at a frequency of 880 kHz and a power of 10kW. How many photons per seconds does it emit.

(Ans: 17.15×10^{30})

8. A mercury arc is rated at 200 W. How many light quanta are emitted per second in radiation having wavelength of 6123Å if the intensity of this line is 2% only. Assume that 50% of power is spent for radiation.

(Ans: 6.16×10^{18})

9. A pulse of radiation consisting of 5×10^4 photons of $\lambda = 3000\text{Å}$ falls on a photosensitive surface whose sensitivity for this wavelength region is $J = 5\text{ mA/W}$. Find the number of photoelectrons liberated by the pulse.

(Ans: 1034)

10. a) How many photons of a radiation of wavelength $\lambda = 5 \times 10^{-7}$ m must fall per second on a blackened metallic plate in order to produce a force of 10^{-5} newton?
b) At what rate will be the temperature of the plate rise if its mass is 1.5 gram and specific heat 0.1? take $h = 6.62 \times 10^{-34}$ J-s.?

(Ans: (a) 7.54×10^{21} b) 4.8×10^3 °C/s)

11. A blue lamp emits light of mean wavelength of 4500 \AA . The lamp is rated a 150W and 8% of the energy appears as emitted light. How many photons are emitted by the lamp per second?

(Ans: 27.15×10^{18})

12. With what velocity must an electron travel so that its momentum is equal to that of a photon with a wavelength of $\lambda = 5200 \text{ \AA}$?

(Ans: $p = 1400$ m/s)

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