Moment of Inertia

1. **Moment of inertia**

   i. Moment of inertia (I) of a body about an axis is defined as the sum of the products of the masses and the squares of their distances of different particles from the axis of rotation.

   ii. For a particle of mass 'm' rotating at a distance r from the axis of rotation. $I = mr^2$

   iii. For a rigid body $I = mk^2$ where K is called radius of gyration

   iv. Effective distance of all particles of the body from the axis of rotation is called radius of gyration. $K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$

   v. MI depends on the mass, distribution of mass, the axis of rotation, shape, size and temperature of the body.

   vi. MI opposes the change in the rotatary motion.

   vii. MI is least about an axis passing through CG of the body.

   viii. MI of any particle on the axis of rotation is zero.

2. Two small spheres of masses m_1 and m_2 are joined by a rod of length 'r' and of negligible mass. The moment of inertia of the system about an axis passing through the centre of mass and perpendicular to the rod, treating the spheres as particles is

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2$$

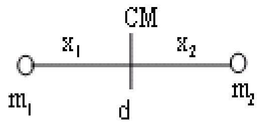
3. **Perpendicular axes theorem:** The moment of inertia of a plane lamina about an axis perpendicular to its plane is the sum of the moments of inertia of the same lamina about two mutually perpendicular axes, lying in the plane of the lamina and intersecting on the given axis.

   $I_z = I_x + I_y$

4. **Parallel axes theorem:** The moment of inertia of any rigid body about any axis is equal to the moment of inertia of the same body about a parallel axis passing through its centre of mass plus the product of the mass of the body and square of the distance between the parallel axes. $I = I_G + Md^2$.

5. If two particles of masses m_1 and m_2 are separated by a distance 'd' then MI of the system of two particles.

1) About CM

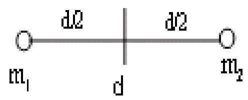


$$x_1 = d\left(\frac{m_2}{m_1+m_2}\right) \text{ And } x_1 = d\left(\frac{m_1}{m_1+m_2}\right)$$

$$\therefore I = m_1x_1^2 + m_2x_2^2$$

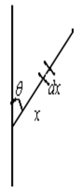
$$I = \left(\frac{m_1m_2}{m_1+m_2}\right)d^2$$

2) About the mid point



$$I = m_1\frac{d^2}{4} + m_2\frac{d^2}{4} = (m_1+m_2)\frac{d^2}{4}$$

7. A uniform rod of m and length l makes a constant angle θ with axis of rotation which passes through one end of the rod. Its MI is $I = \frac{ml^2}{3}\sin^2\theta$



8. Pure rolling: When a body is rolling on a horizontal surface

a. Translational $KE = \frac{1}{2}mv^2$

b. Rotational $KE = \frac{1}{2}I\omega^2 = \frac{1}{2}mK^2\frac{v^2}{r^2} = \frac{1}{2}mv^2\left(\frac{k^2}{r^2}\right)$

c. Total KE = $\frac{1}{2}mv^2\left(1+\frac{K^2}{r^2}\right)$

d. Fractional translation KE = $\frac{r^2}{r^2+k^2}$

e. Fractional rotational KE = $\frac{k^2}{r^2+k^2}$

f. Ratio of translational and rotational $KE = \frac{r^2}{k^2}$

9. **Body rolling on a smooth inclined plane**

a) Acceleration = $a = \dfrac{g\sin\theta}{1+\dfrac{k^2}{r^2}}$

b) Velocity at the bottom $v = \sqrt{\dfrac{2gh}{1+\dfrac{k^2}{r^2}}}$ and

c) Time taken by the body to reach the bottom of the plane $t=\sqrt{\dfrac{2l\left(1+\dfrac{k^2}{r^2}\right)}{g\sin\theta}}=\sqrt{\dfrac{2h\left(1+\dfrac{k^2}{r^2}\right)}{g\sin^2\theta}}$

Body	MI	V	a	t	$\dfrac{E_r}{E_t}$	$\dfrac{E_t}{E}$	$\dfrac{E_r}{E}$
Ring (or) Hollow cylinder	mr^2	$\sqrt{gh}$	$\frac{1}{2}g\sin\theta$	$\sqrt{\dfrac{4l}{g\sin\theta}}$	1 : 1	1 : 2	1 : 2
Hollow sphere	$\frac{2}{3}mr^2$	$\sqrt{\dfrac{6gh}{5}}$	$\frac{1}{3}g\sin\theta$	$\sqrt{\dfrac{10l}{3g\sin\theta}}$	2 : 3	3 : 5	2 : 5
Solid sphere	$\frac{2}{5}mr^2$	$\sqrt{\dfrac{10gh}{7}}$	$\frac{5}{7}g\sin\theta$	$\sqrt{\dfrac{14l}{5g\sin\theta}}$	2 : 5	5 : 7	2 : 7
Disc (or) solid cylinder	$\dfrac{mr^2}{2}$	$\sqrt{\frac{4}{3}gh}$	$\frac{2}{3}g\sin\theta$	$\sqrt{\dfrac{3l}{g\sin\theta}}$	1 : 2	2 : 3	1 : 3

10. A rod of length l is allowed to fall placing it vertical on a table .The lower end which is in contact with the table does not slide.

a) The angular velocity of the tip of the rod when the rod makes an angle θ with the vertical $\omega=\sqrt{\dfrac{3g(1-\cos\theta)}{l}}=\sqrt{\dfrac{6g}{l}}\,Sin\left(\dfrac{\theta}{2}\right)$

b) The linear velocity is given by $v=r\omega=\sqrt{3gl(1-\cos\theta)}$

16. Formulae for moment of inertia for some important cases

Object	**Axis of rotation**	**Moment of inertia**
1. Disc of radius R	1) through its centre and perpendicular to its plane	$\dfrac{MR^2}{2}$
	2) about the diameter	$\dfrac{MR^2}{4}$
	3) about a tangent to its own plane	$\dfrac{5MR^2}{4}$
	4) tangent perpendicular to the plane of the disc	$\dfrac{3MR^2}{2}$
2. Annular ring or disc of outer	1) through its centre and	$\dfrac{M(R^2+r^2)}{2}$

and inner radii R and r	perpendicular to its plane 2) about the diameter 3) about a tangent to its own plane	$\frac{M(R^2+r^2)}{4}$ $\frac{M(5R^2+r^2)}{4}$
3. Solid cylinder of length L and radius R	1) axis of cylinder 2) through its centre and perpendicular to the axis of cylinder 3) diameter of the face	$\frac{MR^2}{2}$ $M\left(\frac{L^2}{12}+\frac{R^2}{4}\right)$ $M\left(\frac{L^2}{3}+\frac{R^2}{4}\right)$
4. Thin rod of uniform length L	1) through its centre and perpendicular to its length 2) through one end and perpendicular to its length	$\frac{ML^2}{12}$ $\frac{ML^2}{3}$
5. Solid sphere of radius R	1) about a diameter 2) about a tangent	$\frac{2}{5}MR^2$ $\frac{7}{5}MR^2$
6. Hollow sphere of radius R	about a diameter	$\frac{2}{3}MR^2$

7. Thin circular ring of radius R	1) Perpendicular to its plane and passing through its centre. 2) about its diameter	MR^2 $\frac{MR^2}{2}$
8. Hollow cylinder of radius R	about axis of the cylinder	MR^2
9. Rectangular lamina of length l and breadth b	1) through its centre and perpendicular to its plane	$M\left(\frac{l^2}{12}+\frac{b^2}{12}\right)$

	2) through its centre and parallel to breadth along its own plane	$\frac{Ml^2}{12}$
	3) through its centre and parallel to length along its own plane	$\frac{Mb^2}{12}$
	4) edge of the length in the plane of the lamina	$\frac{Mb^2}{3}$
	5) edge of the breadth in the plane of the lamina	$\frac{Ml^2}{3}$

11. Angular momentum ($\vec{L}$)

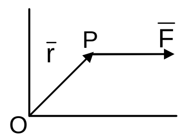
i. The moment of linear momentum is called angular momentum of the particle about the axis of rotation.

ii. $\vec{L} = mvr = mr^2\omega = I\omega$ Or $\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

iii. It is a vector quantity and its SI unit is kgm^2s^{-1}

12. Torgue

i. A force $\bar{F}$ acting on a particle at p whose position vector is $\bar{r}$. Then the torque $\bar{F}$ about 'O' is defined as $\bar{\tau} = \bar{r} x \bar{F} =$



ii. It is an axial vector. Its direction is given by right hand thumb rule. Its S.I unit is N.m

13. Angular impulse ($\vec{J}$)

i. It is the product of torque and time for which it acts.

ii. Angular impulse $= \vec{J} = \vec{\tau} \times t = I\vec{\alpha}t = I\vec{\omega}_2 - I\vec{\omega}_1 = \vec{L}_2 - \vec{L}_1 =$ change in angular momentum

14. Couple

i. Two equal and opposite forces not having the same line of action constitute a couple.

e.g.: Turning water tap, turning the key in a lock.

ii. The moment of couple or torque is the product of one of the forces and the perpendicular distance of separation between the forces.

iii. To balance a couple, another equal but opposite couple is necessary.

15. Law of conservation of angular momentum: When the resultant external torque on a system is zero, the angular momentum of the system remains constant. $I_1\omega_1 = I_2\omega_2 = \ldots =$ constant. Circus acrobats, divers and ballet dancers take advantage of this principle.