MATHEMATICAL METHODS

NUMERICAL SOLUTIONS OF ALGEBRAIC AND TRANSDENTIAL EQUATIONS

I YEAR B.Tech

By

Mr. Y. Prabhaker Reddy
Asst. Professor of Mathematics
Guru Nanak Engineering College
Ibrahimpatnam, Hyderabad.
# SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

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UNIT-IV (a)
SOLUTIONS OF NON-LINEAR SYSTEMS

a) Numerical Solutions of Algebraic and Transcendental Equation
   ➢ Introduction
   ➢ Bisection Method
   ➢ Regular Folsi Method
   ➢ Newton Raphson Method
   ➢ Iteration Method
Aim: To find a root of \( f(x) = 0 \)

**Algebraic Equation**: An Equation which contains algebraic terms is called as an algebraic Equation.

*Example*: \( x^2 + x + 1 = 0 \), Here Highest power of \( x \) is finite. So it is an algebraic Equation.

**Transcendental Equation**: An equation which contains trigonometric ratios, exponential function and logarithmic functions is called as a Transcendental Equation.

*Example*: \( e^x + 2 = 0 \), \( \sin x + 1 = 0 \), \( \log(1 + x) = 0 \) etc.

In order to solve above type of equations following methods exist

- **Directive Methods**: The methods which are used to find solutions of given equations in the direct process is called as directive methods.
  
  *Example*: Synthetic division, remainder theorem, Factorization method etc

  *Note*: By using Directive Methods, it is possible to find exact solutions of the given equation.

- **Iterative Methods (Indirect Methods)**: The methods which are used to find solutions of the given equation in some indirect process is called as Iterative Methods

  *Note*: By using Iterative methods, it is possible to find approximate solution of the given equation and also it is possible to find single solution of the given equation at the same time.

To find a root of the given equation, we have following methods

- Bisection Method
- The Method of false position (Or) Regular false Method
- Iteration Method (Successive approximation Method)
- Newton Raphson Method.
Bisection Method

Consider \( f(x) = 0 \) be the given equation.

Let us choose the constants \( a \) and \( b \) in such a way that \( f(a) \cdot f(b) < 0 \)

(i.e. \( f(a) \) and \( f(b) \) are of opposite signs) i.e. \( f(a) < 0, f(b) > 0 \)

\[(or)\quad f(a) > 0, f(b) < 0\]

Then the curve \( y = f(x) \) crosses \( x\)-axis in between \( a \) and \( b \), so that there exists a root of the given equation in between \( a \) and \( b \).

Let us define initial approximation \( x_0 = \frac{a+b}{2} \)

Now, If \( f(x_0) = 0 \) then \( x_0 \) is root of the given equation.

If \( f(x_0) \neq 0 \) then either \( f(x_0) < 0 \) (or) \( f(x_0) > 0 \)

**Case (i):** Let us consider that \( f(x_0) < 0 \) and \( f(a) > 0 \)

Since \( f(x_0) \cdot f(a) < 0 \)

\[\Rightarrow \text{Root lies between } x_0 \text{ and } a\]

\[\Rightarrow x_1 = \frac{x_0 + a}{2}\]

Here \( f(x_1) < 0 \) (or) \( f(x_1) > 0 \)

Let us consider that \( f(x_1) < 0 \& f(a) > 0 \)

Since \( f(x_1) \cdot f(a) < 0 \)

\[\Rightarrow \text{Root lies between } x_0 \text{ and } a\]

\[\Rightarrow x_2 = \frac{x_1 + a}{2}\]

Here, the logic is that, we have to select first negative or first positive from bottom approximations only, but not from top. i.e. the approximation which we have recently found should be selected.

If \( f(x_1) = 0 \), then \( x_1 \) is root of the given equation. Otherwise repeat above process until we obtain solution of the given equation.
Regular Folsi Method (Or) Method of False position

Let us consider that \( y = f(x) \) be the given equation.

Let us choose two points \( a \) and \( b \) in such a way that \( f(a) \) and \( f(b) \) are of opposite signs.

i.e. \( f(x_0). f(x_1) < 0 \)

Consider \( f(a) > 0, f(b) < 0 \), so that the graph \( y = f(x) \) crosses \( X \)-axis in between \( a \) and \( b \). then, there exists a root of the given equation in between \( a \) and \( b \).

Let us define a straight line joining the points \( A(a, f(a)) \), \( B(b, f(b)) \), then the equation of straight line is given by

\[
\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a} \quad \text{---- (I)}
\]

This method consists of replacing the part of the curve by means of a straight line and then takes the point of intersection of the straight line with \( X \)-axis, which give an approximation to the required root.

In the present case, it can be obtained by substituting \( y = 0 \) in equation I, and it is denoted by \( x_0 \).

From (I) \( \Rightarrow y - f(a) = \frac{f(b)-f(a)}{b-a} (x - a) \)

Now, any point on \( X \)-axis \( \Rightarrow y = 0 \) and let initial approximation be \( x_0 \) i.e. \( x = x_0 \)

\( \therefore \) (I) \( \Rightarrow 0 - f(a) = \frac{f(b)-f(a)}{b-a} (x_0 - a) \)

\( \Rightarrow x_0 - a = \frac{-f(a)(b-a)}{f(b)-f(a)} \)

\( \Rightarrow x_0 = a - \frac{f(a)(b-a)}{f(b)-f(a)} \)

\( \Rightarrow x_0 = a - \frac{f(b)-f(a)}{f(b)-f(a)} \)

\( \Rightarrow x_0 = \frac{a f(b)-b f(a)}{f(b)-f(a)} \quad \text{---- (II)}\)

If \( f(x_0) = 0 \), then \( x_0 \) is root of the given equation.

If \( f(x_0) \neq 0 \), then either \( f(x_0) < 0 \) (or) \( f(x_0) > 0 \)

**Case (i):** Let us consider that \( f(x_0) < 0 \)

We know that \( f(a) > 0 \) and \( f(x_0) < 0 \)

Hence root of the given equation lies between \( a \) and \( x_0 \).
In order to obtain next approximation \( x_1 \), replace \( b \) with \( x_0 \) in equation (II)

Hence \( x_1 = \frac{a f(x_0) - x_0 f(a)}{f(x_0) - f(a)} \)

**Case (ii):** Let us consider that \( f(x_0) > 0 \)

We know that \( f(b) < 0 \) and \( f(x_0) > 0 \)

Hence root of the given equation lies between \( b \) and \( x_0 \).

In order to obtain next approximation \( x_2 \), replace \( a \) with \( x_0 \) in equation (II)

Hence \( x_2 = \frac{x_0 f(b) - b f(x_0)}{f(b) - f(x_0)} \)

If \( f(x_2) = 0 \), then \( x_2 \) is root of the given equation. Otherwise repeat above process until we obtain a root of the given equation to the desired accuracy.

**Newton Raphson Method**

Let us consider that \( f(x) = 0 \) be the given equation.

Let us choose initial approximation to be \( x_0 \).

Let us assume that \( x_1 = x_0 + h \) be exact root of \( f(x) = 0 \) where \( h \ll 0 \), so that \( f(x_0 + h) = 0 \)

Expanding above relation by means of Taylor’s expansion method, we get

\[
f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots = 0
\]

Since \( h \) is very small, \( h \) and higher powers of \( h \) are neglected. Then the above relation becomes

\[
f(x_0) + h f'(x_0) = 0 \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}
\]

Hence \( x_1 = x_0 + h \)

\[
\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\]

If \( f(x_1) = 0 \), then \( x_1 \) is root of the given equation. Otherwise repeat above process until we obtain a root of the given equation to the desired accuracy.

Successive approximations are given by

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad \text{and so on,} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]
Iteration Method

Let us consider that \( f(x) = 0 \) be the given equation.

Let us choose initial approximation to be \( x_0 \).

Rewrite \( f(x) = 0 \) as \( x = \varphi(x) \) such that \( |\varphi'(x_0)| < 0 \).

Then successive approximations are given by \( x_1 = \varphi(x_0) \)

\[
\begin{align*}
x_2 &= \varphi(x_1) \\
\vdots \\
x_n &= \varphi(x_{n-1})
\end{align*}
\]

Repeat the above process until we get successive approximation equal, which will gives the required root of the given equation.